Impedance tomography in anisotropic media

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Calderón problem

Medical imaging, Electrical Impedance Tomography:

$$\begin{cases} \operatorname{div}(\gamma(x)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

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Boundary measurements given by DN map

$$\Lambda_\gamma : f \mapsto \gamma \nabla u \cdot \nu|_{\partial\Omega}.$$

Inverse problem: given Λ_γ, determine γ.

Calderón problem

Major results:

- Calderón (1980): linearized problem
- Sylvester-Uhlmann (1987): $n \geq 3$, $\gamma \in C^\infty(\overline{\Omega})$
- Nachman (1996): $n = 2$, $\gamma \in W^{2,p}(\Omega)$
- Astala-Päivärinta (2006): $n = 2$, $\gamma \in L^\infty(\Omega)$

Anisotropic problem

We are interested in the anisotropic case, where

$$\gamma(x) = (\gamma^{jk}(x))_{j,k=1}^{n}$$

is a symmetric positive definite matrix.

The conductivity of the medium depends on the direction.
This is relevant in applications (e.g. imaging muscle).

Anisotropic problem

Dirichlet problem

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$$\Lambda_\gamma : f \mapsto \gamma \nabla u \cdot \nu|_{\partial\Omega}.$$

Inverse problem: given Λ_γ, determine $\gamma = (\gamma^{jk})$.

Obstruction

There is an obstruction to uniqueness. If $F : \Omega \to \Omega$ is a diffeomorphism with $F|_{\partial\Omega} = \mathrm{id}_{\partial\Omega}$, then

$$\Lambda_{F_*\gamma} = \Lambda_\gamma.$$

Here $F_*\gamma$ is the pushforward

$$F_*\gamma(y) = \frac{DF \circ \gamma \circ (DF)^t}{\det(DF)}\bigg|_{F^{-1}(y)}.$$

Anisotropic problem

Conjecture 1. Let $\gamma_1, \gamma_2 \in C^\infty(\overline{\Omega})$ be two symmetric positive definite matrices. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then

$$\gamma_2 = F_* \gamma_1$$

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- Sylvester (1990): $n = 2$, small conductivities
- Nachman (1996): $n = 2$, $\gamma \in W^{2,p}$
- Astala-Lassas-Päivärinta (2005): $n = 2$, $\gamma \in L^\infty$

For $n \geq 3$ this is an important open problem.

Geometric problem

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Let (M, g) be a compact smooth Riemannian manifold with boundary ∂M. The Laplace-Beltrami operator Δ_g on M is given by

$$\Delta_g u = \sum_{j,k=1}^{n} \frac{1}{\sqrt{\det\ g}} \frac{\partial}{\partial x_j} \left(\sqrt{\det\ g}\, g^{jk} \frac{\partial u}{\partial x_k} \right),$$

where $g = (g_{jk}), g^{-1} = (g^{jk})$.

Geometric problem

Consider Dirichlet problem

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{cases}$$

Boundary measurements

$$\Lambda_g : f \mapsto \partial_\nu u|_{\partial M}.$$

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Similar obstruction as for conductivity. If $F : M \longrightarrow M$ is a diffeomorphism with $F|_{\partial M} = \mathrm{id}_{\partial M}$, then $\Lambda_{F^*g} = \Lambda_g$.

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Similar obstruction as for conductivity. If $F : M \to M$ is a diffeomorphism with $F|_{\partial M} = \mathrm{id}_{\partial M}$, then $\Lambda_{F^* g} = \Lambda_g$.

Conjecture 2. Let (M, g_1) and (M, g_2) compact smooth manifolds with boundary. If $\Lambda_{g_1} = \Lambda_{g_2}$, then

$$g_2 = F^* g_1$$

where $F : M \longrightarrow M$ diffeomorphism with $F|_{\partial M} = \mathrm{id}_{\partial M}$.

That is, does Λ_g determine (M, g) up to isometry?

Results

Known results on Conjecture 2 if $n \geq 3$:

- Lee-Uhlmann (1989): g real-analytic
- Lassas-Uhlmann (2001), Lassas-Taylor-Uhlmann (2003): g real-analytic, removed topological assumptions
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These are based on boundary determination and analyticity.

Complex geometrical optics

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Our results are based on an extension of the complex geometrical optics method to a class of Riemannian manifolds.

To do this, we adapt the Carleman estimate approach of Kenig-Sjöstrand-Uhlmann (2007) to geometric setting.

Limiting Carleman weights

Need complex geometrical optics solutions

$$u = e^{\tau(\varphi + i\psi)}(a + r) \qquad \text{(cf. } u = e^{\rho \cdot x}(1 + r)\text{)}$$

to $\Delta_g u = 0$.

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to $\Delta_g u = 0$.

Here τ is a large parameter, and φ is a *limiting Carleman weight* (LCW): the Carleman estimate

$$\|e^{\tau\varphi} u\|_{L^2(M)} \leq \frac{C}{\tau} \|e^{\tau\varphi} \Delta_g u\|_{L^2(M)},$$

holds both for φ and $-\varphi$.

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holds both for φ and $-\varphi$.

Examples in $\mathbf{R}^n$: $\varphi(x) = x_1$ and $\varphi(x) = \log|x|$.

Characterization

Theorem 1. A simply connected manifold (M, g) admits an LCW if and only if it is *conformally transversally anisotropic*.

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Examples of such manifolds:

1. bounded domains in $\mathbf{R}^n$, $S^n \setminus \{p_0\}$, H^n
2. conformally flat manifolds, e.g. 3D symmetric spaces
3. bounded domains (Ω, g) in $\mathbf{R}^n$ where

$$g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}.$$

Euclidean case

In Euclidean space, we can characterize all LCWs.

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Theorem 2. If φ is an LCW in (Ω, e) where $\Omega \subseteq \mathbf{R}^n$, $n \geq 3$, then

$$\varphi(x) = a\varphi_0(x - x_0) + b,$$

where φ_0 is one of the following:

$$x \cdot \xi, \quad \log|x|, \quad \frac{x \cdot \xi}{|x|^2}, \quad \arctan\frac{x \cdot \xi}{x \cdot \eta},$$
$$\arctan\frac{2x \cdot \xi}{|x|^2 - |\xi|^2}, \quad \operatorname{arctanh}\frac{2x \cdot \xi}{|x|^2 + |\xi|^2}.$$

Recovering coefficients

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To ensure injectivity of certain geodesic ray transforms, we need another condition on the manifold.

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Theorem. Let (M, g) be simple and $f \in C^\infty(M)$. Suppose that

$$\int_\gamma f(\gamma(t))\, dt = 0$$

for all geodesics γ going from ∂M into M. Then $f = 0$.

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Theorem 3. Let (M, g) be simple, $a \in C^\infty(M)$ sufficiently small, and $f \in C^\infty(M)$. Suppose that

$$\int_\gamma f(\gamma(t)) \exp\left[\int_0^t a(\gamma(s))\,ds\right] dt = 0$$

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Definition. A compact manifold (M, g) with boundary is called *simple* if it has no conjugate points, and ∂M is strictly convex.

Theorem 3'. Let (M, g) be simple, $a \in C^\infty(M)$ and $n = 2$, and $f \in C^\infty(M)$. Suppose that

$$\int_\gamma f(\gamma(t)) \exp\left[\int_0^t a(\gamma(s))\, ds\right] dt = 0$$

for all geodesics γ going from ∂M into M. Then $f = 0$.

Joint work with Gunther Uhlmann (2010).

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with g_0 simple.

The class of admissible manifolds is stable under small perturbations of g_0.

Recovering a potential

Consider Dirichlet problem

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Theorem 4. Let (M, g) be admissible and $q_1, q_2 \in C^\infty(M)$. If $\Lambda_{g,q_1} = \Lambda_{g,q_2}$, then $q_1 = q_2$.

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Also possible to recover a magnetic field.

Recovering a conformal factor

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Theorem 5. Let (M, g_1) and (M, g_2) be admissible manifolds in the same conformal class. If $\Lambda_{g_1} = \Lambda_{g_2}$, then $g_1 = g_2$.

Maxwell equations

Consider the Maxwell equations in $\Omega \subseteq \mathbf{R}^3$,

$$\begin{cases} \nabla \times E = i\omega\mu H, \\ \nabla \times H = -i\omega\varepsilon E \end{cases}$$

with boundary condition

$$E_{\tan}|_{\partial\Omega} = f.$$

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$$\Lambda_{\varepsilon,\mu} : f \mapsto H_{\tan}|_{\partial\Omega}.$$

Maxwell equations

Theorem 6. (KSU 2009) Let ε and μ be 2-tensors conformal to

$$\begin{pmatrix} 1 & 0 \\ 0 & g_0(x')^{-1} \end{pmatrix}$$

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Earlier results:

- Ola-Päivärinta-Somersalo (1993): scalar ε and μ
- Greenleaf-Kurylev-Lassas-Uhlmann (2007): nonuniqueness (invisibility) for exotic ε and μ