1 Introduction

These notes follow chapter 6 of [2].

2 Stopping time

Definition 2.1 (*Stopping time*). *A random variable*

$$\tau\colon \Omega \to [0\,,\infty]$$

is called a stopping time with respect to a filtration of σ-algebras $\{\mathcal{F}_t\,|\,t \geq 0\}$ provided

$$\{\tau \leq t\} \in \mathcal{F}_t \qquad\qquad \textit{for all} \quad t \geq 0$$

In other words, the set of all $\omega \in \Omega\ \tau(\omega) \leq t$ is $\mathcal{F}_t$-measurable. The stopping time τ is allowed to take on the value $+\infty$, and also that any constant $\tau = t_o$ is a stopping time. Furthermore it enjoys the following properties

Proposition 2.1 (*Properties of a stopping time*). *Let τ_1 and τ_2 stopping times with respect to $\{\mathcal{F}_t\,|\,t \geq 0\}$. Then*

i $\{\tau < t\} \in \mathcal{F}_t$ *and* $\{\tau = t\} \in \mathcal{F}_t$ *for all* $t \geq 0$

ii $\tau_1 \wedge \tau_2$ *and* $\tau_1 \vee \tau_2$ *are stopping times*

Proof. We set

$$\{\tau < t\} = \bigcup_{k=1}^{\infty} \left\{\tau \leq t - \frac{1}{k}\right\}$$

i.e. $\{\tau < t\}$ occurs if there exists a $k \geq 1$ such that the event $\{\tau \leq t - 1/k\}$ occurs. But

$$\{\tau \leq t - 1/k\} \in \mathcal{F}_{t-\frac{1}{k}} \subseteq \mathcal{F}_t$$

Similarly

$$\{\tau_1 \wedge \tau_2 \leq t\} = \{\tau_1 \leq t\} \cup \{\tau_2 \leq t\} \in \mathcal{F}_t$$

and

$$\{\tau_1 \vee \tau_2 \leq t\} = \{\tau_1 \leq t\} \cap \{\tau_2 \leq t\} \in \mathcal{F}_t$$

□

The following theorem evinces the relevance of stopping times for the study of stochastic differential equations

Theorem 2.1 (*Set hitting by a diffusion*). *Let $\boldsymbol{\xi}_t$ solution of the stochastic differential equation*

$$d\boldsymbol{\xi}_t = \boldsymbol{b}\,(\boldsymbol{\xi}_t, t)\,dt + \boldsymbol{\sigma}[d\boldsymbol{w}_t]$$

$$\boldsymbol{\xi}_{t_o} = \boldsymbol{x}_o$$

satisfying the hypotheses of the theorem of existence and uniquensess. Let also $\mathbb{A}$ be a non-empty open or closed subset of $\mathbb{R}^d$. Then

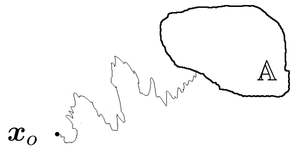
$$\tau := \inf\,\{t\,|\,\xi_t \in \mathbb{A}\}$$

is a stopping time with the convention $\tau = \infty$ if $\boldsymbol{\xi}_t \notin \mathbb{A}$ for all t.

Proof. Let $t \geq 0$ we need to show that $\{\tau \leq t\} \in \mathcal{F}_t$. To that goal we introduce the sequence $\{t_i\}_{i=1}^{\infty}$ dense on $\mathbb{R}_+$ and consider separately the cases when $\mathbb{A}$ is close and open.

- $\mathbb{A}$ is open. The event that there exists a t_i less than t such that $\boldsymbol{\xi}_{t_i}$ belongs to $\mathbb{A}$ reads

$$\{\tau \leq t\} = \bigcup_{t_i \leq t} \{\boldsymbol{\xi}_{t_i} \in \mathbb{A}\}$$



and is therefore the union of events belonging to $\mathcal{F}_t$, thus proving the claim.

- $\mathbb{A}$ is closed. Let

$$d\,(\boldsymbol{x}, \mathbb{A}) := \text{distance}\,(\boldsymbol{x}, \mathbb{A})$$

and define the open sets

$$\mathbb{A}_n = \left\{ \boldsymbol{x} \,|\, d(\boldsymbol{x}\,, \mathbb{A}) < \frac{1}{n} \right\}$$

The event

$$\{\tau \leq t\} = \bigcap_{k=1}^{\infty} \bigcup_{t_i \leq t} \{\boldsymbol{\xi}_{t_i} \in \mathbb{A}_n\}$$

also belongs to $\mathcal{F}_t$ as the $\left\{\boldsymbol{\xi}_{t_i} \in \mathbb{A}_n\right\}$'s do.

□

Remark 2.1. The random variable

$$\tilde{\tau} = \sup\,\{t \,|\, \xi_t \in \mathbb{A}\}$$

is not in general a stopping time as in general it is not $\mathcal{F}_t$ measurable but may depend on the history of $\boldsymbol{\xi}$ for times later than t.

3 Applications of the stopping time

Let $\boldsymbol{\phi}_t$ be the fundamental solution of the stochastic differential equation

$$d\xi_t^i = b^i\,(\boldsymbol{\xi}_t\,, t)\;dt + \sigma^{ij}\,(\boldsymbol{\xi}_t\,, t)\;dw_t^j \tag{3.1}$$

which we assume to globally satisfy the hypotheses of the theorem existence and uniqueness of solutions. In other words for any initial data $(\boldsymbol{x}_o\,, t_o)$ we have that

$$\boldsymbol{\xi}_t = \boldsymbol{\phi}_t\,(\boldsymbol{x}_o\,, t_o)$$

for $t \geq t_o$ solves (3.1). To (3.1) also we associate the generator

$$\mathfrak{L}_{\boldsymbol{x}} := \boldsymbol{b}\,(\boldsymbol{x}, t) \cdot \partial_{\boldsymbol{x}} + \frac{1}{2} g^{ij}\,(\boldsymbol{x}, t)\,\partial_{x^i}\partial_{x^j}$$

with

$$g^{ij} = \sigma^{ik}\sigma^{jk}$$

3.1 Exit time form a domain

Let $\mathbb{A}$ a smooth bounded open subset of $\mathbb{R}^d$ Let us suppose that the drift and diffusion fields in (3.1) are time-independent

$$\boldsymbol{b}(\boldsymbol{x},t)=\boldsymbol{b}(\boldsymbol{x}) \qquad \& \qquad \boldsymbol{\sigma}(\boldsymbol{x},t)=\boldsymbol{\sigma}(\boldsymbol{x})$$

and let $\boldsymbol{x}\in\mathbb{R}^d$. Then the stopping time

$$\tau_{\boldsymbol{x}}=\inf\{t\geq\,0|\boldsymbol{\phi}_t(\boldsymbol{x}\,,0)\in\partial\mathbb{A}\} \tag{3.2}$$

specifies the first exit time from $\mathbb{A}$.

Proposition 3.1 (*Average exit time*). *Under the above hypotheses, for any $\boldsymbol{x}\in\mathbb{A}$ we have*

$$\prec\tau_{\boldsymbol{x}}\succ=f(\boldsymbol{x})$$

for

$$\mathfrak{L}_{\boldsymbol{x}}f(\boldsymbol{x})=-1$$

$$f(\boldsymbol{x})\,|_{\boldsymbol{x}\in\mathbb{A}}=0$$

More generally for we have

$$\prec\tau_{\boldsymbol{x}}^n\succ=g_n(\boldsymbol{x})$$

for $g_0(\boldsymbol{x})=1$

$$\mathfrak{L}_{\boldsymbol{x}}g_n(\boldsymbol{x})=-n\,g_{n-1}(\boldsymbol{x})$$

$$g_n(\boldsymbol{x})\,|_{\boldsymbol{x}\in\mathbb{A}}=0$$

Proof. Let

$$f\left(\boldsymbol{\phi}_{\tau_{\boldsymbol{x}}}\right)=f(\boldsymbol{x})+\int_0^{\tau_{\boldsymbol{x}}}ds\,\mathfrak{L}_{\boldsymbol{\phi}_s}f(\boldsymbol{\phi}_s)+\int_0^{\tau_{\boldsymbol{x}}}dw_s^j\,\sigma^{ij}(\boldsymbol{\phi}_s)\,\partial_{\phi_s^i}f(\boldsymbol{\phi}_s)$$

if

$$\mathfrak{L}_{\boldsymbol{\phi}_s}f(\boldsymbol{\phi}_s)=-1$$

and

$$f(\boldsymbol{x})\,|_{\boldsymbol{x}\in\mathbb{A}}=0$$

we have

$$\tau_{\boldsymbol{x}}=f(\boldsymbol{x})+\int_0^{\tau_{\boldsymbol{x}}}dw_s^j\,\sigma^{ij}(\boldsymbol{\phi}_s)\,\partial_{\phi_s^i}f(\boldsymbol{\phi}_s)$$

Taking averages yields the claim. More generally for any $t\leq\tau_{\boldsymbol{x}}$

$$g_n(\boldsymbol{\phi}_t)=-\int_t^{\tau_{\boldsymbol{x}}}ds\,\mathfrak{L}_{\boldsymbol{\phi}_s}g_n(\boldsymbol{\phi}_s)-\int_t^{\tau_{\boldsymbol{x}}}dw_s^j\,\sigma^{ij}(\boldsymbol{\phi}_s)\,\partial_{\phi_s^i}g_n(\boldsymbol{\phi}_s)$$

$$\mathfrak{L}_{\boldsymbol{x}} g_n(\boldsymbol{x}) = -n\, g_{n-1}(\boldsymbol{x})$$

$$g_{n-1}(\boldsymbol{x})\,|_{\boldsymbol{x}\in\partial\mathbb{A}} = 0$$

generates the recursion for

$$g_n(\boldsymbol{\phi}_0) = g_n(\boldsymbol{x})$$

whence

$$\begin{aligned} g_n(\boldsymbol{x}) &= n \int_0^{\tau_{\boldsymbol{x}}} ds \int_s^{\tau_{\boldsymbol{x}}} ds_1 \mathfrak{L}_{\boldsymbol{\phi}_{s_1}} g_{n-1}(\boldsymbol{\phi}_{s_1}) \\ &+ n \int_0^{\tau_{\boldsymbol{x}}} ds \int_s^{\tau_{\boldsymbol{x}}} dw_{s_1} \sigma^{ij}(\boldsymbol{\phi}_{s_1})\, \partial_{\phi^i_{s_1}} g_{n-1}(\boldsymbol{\phi}_{s_1}) - \int_0^{\tau_{\boldsymbol{x}}} dw_s^j\, \sigma^{ij}(\boldsymbol{\phi}_s)\, \partial_{\phi^i_s} g_n(\boldsymbol{\phi}_s) \end{aligned}$$

Upon iterating n-times, we get into

$$\begin{aligned} g_n(\boldsymbol{x}) &= \Gamma(n+1) \int_0^{\tau_{\boldsymbol{x}}} ds_1 \prod_{k=1}^{l-2} \int_{s_k}^{\tau_{\boldsymbol{x}}} ds_{k+1} \int_{s_{n-1}}^{\tau_{\boldsymbol{x}}} ds_n \\ &- \sum_{l=1}^{n} \frac{\Gamma(n+1)}{\Gamma(n-l+1)} \int_0^{\tau_{\boldsymbol{x}}} ds \prod_{k=1}^{l-2} \int_0^{s_k} ds_{k+1} \int_{s_{l-1}}^{\tau_{\boldsymbol{x}}} dw_{s_l} \sigma^{ij}(\boldsymbol{\phi}_{s_l})\, \partial_{\phi^i_{s_l}} g_{n-l}(\boldsymbol{\phi}_{s_l}) - \int_0^{\tau_{\boldsymbol{x}}} dw_s^j\, \sigma^{ij}(\boldsymbol{\phi}_s)\, \partial_{\phi^i_s} g_n(\boldsymbol{\phi}_s) \end{aligned}$$

Taking the average we get into

$$\prec \tau_{\boldsymbol{x}}^n \succ = g_n(\boldsymbol{x})$$

whence the claim. □

In order to encompass the time-inhomogeneous case, we can compute the statistics of the first exit time from an open subset of $\mathbb{R}^d$ starting from the forward Kolmogorov equation (Fokker-Planck equation). Consider for any $\boldsymbol{x}_o \in \mathbb{A}$ the problem with *absorbing boundary conditions*

$$\partial_t p + \partial_{x^i}(\, b^i\, p) = \frac{1}{2}\partial_{x^i}\partial_{x^j}(g^{ij} p) \tag{3.6a}$$

$$p|_{\boldsymbol{x}\in\partial\mathbb{A}} = 0 \tag{3.6b}$$

$$\lim_{t\downarrow t_o} p = \delta^{(d)}(\boldsymbol{x} - \boldsymbol{x}_o) \tag{3.6c}$$

The interpretation of absorbing boundary conditions is of removing from the transition probability all those trajectories that for times $s \in [t_o, t]$ reached the boundary. Thus if we define

$$\tau_{\boldsymbol{x},t} = \inf_{t_1} \left\{ t \le t_1 | \boldsymbol{\phi}_{t_1}(\boldsymbol{x}\,, t) \in \partial\mathbb{A} \right\} \tag{3.7}$$

we get into

$$P(\tau_{\boldsymbol{x}_o,t_o} \ge t) = \int_{\mathbb{A}} d^d x\, p_{\boldsymbol{\xi}}(\boldsymbol{x}, t\,|\,\boldsymbol{x}_o, t_o)$$

whence we infer

$$p_{\tau_{\boldsymbol{x}_o,t_o}}(t) = -\partial_t \int_{\mathbb{A}} d^d x \, p_{\boldsymbol{\xi}} \left(\boldsymbol{x}, t \,|\, \boldsymbol{x}_o, t_o \right)$$

It follows immediately that

$$\prec (\tau_{\boldsymbol{x}_o,t_o} - t_o)^n \succ = \int_{t_o}^{\infty} dt \, (t - t_o)^n \, p_{\tau_{\boldsymbol{x}_o,t_o}}(t) = -\int_0^{\infty} dt \, t^n \, \partial_t \int_{\mathbb{A}} d^d x \, p_{\boldsymbol{\xi}} \left(\boldsymbol{x}, t + t_o \,|\, \boldsymbol{x}_o, t_o \right)$$

and therefore

$$\prec (\tau_{\boldsymbol{x}_o} - t_o)^n \succ = n \int_0^{\infty} dt \, t^{n-1} \int_{\mathbb{A}} d^d x \, p_{\boldsymbol{\xi}} \left(\boldsymbol{x}, t + t_o \,|\, \boldsymbol{x}_o, t_o \right) = g_n(\boldsymbol{x}_o, t_o) \tag{3.8}$$

for χ the characteristic function of the set $\mathbb{A}$. We can derive the evolution equation for the average by differentiating with respect to t_o:

$$\partial_{t_o} g_n(\boldsymbol{x}_o, t_o) = n \int_0^{\infty} dt \, t^{n-1} \left(\partial_t - \mathfrak{L}_{\boldsymbol{x}_o} \right) \int_{\mathbb{A}} d^d x \, p_{\boldsymbol{\xi}} \left(\boldsymbol{x}, t + t_o \,|\, \boldsymbol{x}_o, t_o \right)$$

Inspection of the result allows us to recognize that

$$\partial_{t_o} g_n(\boldsymbol{x}_o, t_o) = -n \, g_{n-1}(\boldsymbol{x}_o, t_o) - \mathfrak{L}_{\boldsymbol{x}_o} g_n(\boldsymbol{x}_o, t_o) \tag{3.9}$$

which is the result we set ut to obtain. Note that the solution of (3.9) can be rewritten as

$$g_n \left(\boldsymbol{x}_o, t_o \right) = \int_0^{\infty} ds \int_{\mathbb{R}^d} d^d x \, n \, g_{n-1} \left(\boldsymbol{x}, s + t_o \right) \, p_{\boldsymbol{\xi}} \left(\boldsymbol{x}, s + t_o | \boldsymbol{x}_o, t_o \right)$$

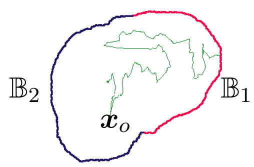
since we can always assume

$$g_n \left(\boldsymbol{x}_o, t_o \right) = 0 \qquad \forall \boldsymbol{x}_o \in \mathbb{R}^d / \mathbb{A} \tag{3.10}$$

3.2 Hitting one part of a boundary first

Suppose now that the boundary $\partial\mathbb{A}$ of a $\mathbb{A}$ a smooth bounded open subset of $\mathbb{R}^d$ can be decomposed as

$$\partial\mathbb{A} = \mathbb{B}_1 + \mathbb{B}_2$$



with $\mathbb{B}_i$ $i = 1, 2$ smooth. To any $\boldsymbol{x}_o \in \mathbb{A}$ we can associate the stopping time

$$\tau_{\mathbb{B}_i|\boldsymbol{x}} = \inf \left\{ t \geq 0 \,|\, \boldsymbol{\phi}_t \left(\boldsymbol{x} \right) \in \mathbb{B}_i \right\} \qquad i = 1, 2 \tag{3.11}$$

through the mapping defined by the fundamental solution of (3.1).

Proposition 3.2. *The probability that $\boldsymbol{\phi}_t(\boldsymbol{x}_o)$ hits first $\mathbb{B}_1$ is specified by the solution of*

$$\mathfrak{L}_{\boldsymbol{x}} \, u \left(\boldsymbol{x} \right) = 0$$

$$u \left(\boldsymbol{x} \right) |_{\boldsymbol{x} \in \mathbb{B}_1} = 1 \qquad \& \qquad u \left(\boldsymbol{x} \right) |_{\boldsymbol{x} \in \mathbb{B}_2} = 0$$

Proof. Let

$$u\left(\boldsymbol{\phi}_t\left(\boldsymbol{x}_o\right)\right)=u\left(\boldsymbol{x}_o\right)+\int_0^t ds\,\mathfrak{L}_{\boldsymbol{\phi}_s}\,u\left(\boldsymbol{\phi}_s\right)+\int_0^t dw_s^j\,\sigma^{ij}\left(\boldsymbol{\phi}_s\right)\partial_{\phi_s^i}u\left(\boldsymbol{\phi}_s\right)$$

If we require

$$\mathfrak{L}_{\boldsymbol{x}}u\left(\boldsymbol{x}\right)=0$$

together with the hypothesized boundary conditions, taking the average yields for $t=\tau_{\mathbb{B}_1:\boldsymbol{x}}$

$$P\left(\boldsymbol{\phi}_{\tau_{\mathbb{B}_1|\boldsymbol{x}_o}}\in\mathbb{B}_1\right)=u\left(\boldsymbol{x}\right)$$

□

3.2.1 Recurrence of the Wiener process

Let $\boldsymbol{w}_t$ a d-dimensional Wiener motion

$$\xi_t=\left|\left|\boldsymbol{w}_t^2\right|\right|$$

then

$$d\xi_t=d\,dt+2\sqrt{\xi_t}\frac{\boldsymbol{w}_t\cdot d\boldsymbol{w}_t}{||\boldsymbol{w}_t||}$$

The stochastic process

$$\eta_t=\int_0^t\frac{\boldsymbol{w}_s\cdot d\boldsymbol{w}_s}{||\boldsymbol{w}_s||}$$

enjoys the following properties

- Vanishing first moment

$$\prec\eta_t\succ=0$$

- Second moment linearly growing in time

$$\prec\eta_t^2\succ=\int_0^t ds=t$$

- Gaussian statistics.
- Independent increments

$$\eta_{t+t_o}-\eta_{t_o}=\int_{t_o}^{t+t_o}\frac{\boldsymbol{w}_s\cdot d\boldsymbol{w}_s}{||\boldsymbol{w}_s||}$$

Hence η_t is statistically equivalent to a Wiener process:

$$d\xi_t=d\,dt+2\sqrt{\xi_t}dw_t$$

We can ask whether the Wiener process leaves a ball of radius R around the origin before hitting the origin itself. To answer such question we need to solve for some $0 < \varepsilon < 1$

$$0 = d\,\partial_x u + 2\,x\partial_x^2 u \tag{3.13a}$$

$$u\,(\varepsilon) = 0 \qquad \& \qquad u\,(R) = 1 \tag{3.13b}$$

A straightforward calculation yields

$$u(x) = P\left(\tau_{\varepsilon|x} \leq \tau_{R|x}\right) = \begin{cases} \dfrac{R^{1-\frac{d}{2}} - x^{1-\frac{d}{2}}}{R^{1-\frac{d}{2}} - \varepsilon^{1-\frac{d}{2}}} & d \neq 2 \\ \dfrac{\ln R - \ln x}{\ln R - \ln \varepsilon} & d = 2 \end{cases}$$

We observe

$$\lim_{\varepsilon \downarrow 0} P\left(\tau_{\varepsilon|x} \leq \tau_{R|x}\right) = \begin{cases} 1 - \left(\dfrac{x}{R}\right)^{1/2} & d = 1 \\ 0 & d \geq 2 \end{cases}$$

In *two dimensions*, nevertherless

$$\lim_{R \uparrow \infty} P\left(\tau_{\varepsilon|x} \leq \tau_{R|x}\right) = 1$$

meaning that the process is *recurrent* in the sense that if $\mathbb{G}$ is any open set

$$P(||\boldsymbol{w}_t||^2 \in \mathbb{G}) = 1$$

Appendix

A Alternative check of the recursion relations for exit times

Iterating the recursion equations as

$$g_n\,(\boldsymbol{x}_o, t_o) = \prod_{i=1}^{2} \int_0^\infty ds_i \int_{\mathbb{R}^d} d^d x_i\, n\,(n-1)\, g_{n-2}\,(\boldsymbol{x}_2, s_2 + s_1 + t_o)\; p_{\boldsymbol{\xi}}\,(\boldsymbol{x}_2, s_2 + s_1 + t_o | \boldsymbol{x}_1, s_1 + t_o)\, p_{\boldsymbol{\xi}}\,(\boldsymbol{x}_1, s_1 + t_o | \boldsymbol{x}_o, t_o)$$

we get by Chapman-Kolmogorov

$$g_n\,(\boldsymbol{x}_o, t_o) = \prod_{i=1}^{2} \int_0^\infty ds_i \int_{\mathbb{R}^d} d^d x_2\, n\,(n-1)\, g_{n-2}\,(\boldsymbol{x}_2, s_2 + s_1 + t_o)\; p_{\boldsymbol{\xi}}\,(\boldsymbol{x}_2, s_2 + s_1 + t_o\,|\,\boldsymbol{x}_o, t_o)$$

Upon setting

$$s = s_1 + s_2$$

the time integral becomes

$$
\begin{aligned}
&g_n\left(\boldsymbol{x}_o, t_o\right) \\
&\quad = \int_0^\infty ds_1 \int_{s_1}^\infty ds \int_{\mathbb{R}^d} d^d x \, n\,(n-1)\, g_{n-2}\left(\boldsymbol{x}, s+t_o\right)\, p_{\boldsymbol{\xi}}\left(\boldsymbol{x}, s+t_o \,|\, \boldsymbol{x}_o, t_o\right) \\
&\quad = n\,(n-1) \int_0^\infty ds\, s \int_{\mathbb{R}^d} d^d x \, g_{n-2}\left(\boldsymbol{x}, s+t_o\right)\, p_{\boldsymbol{\xi}}\left(\boldsymbol{x}, s+t_o \,|\, \boldsymbol{x}_o, t_o\right)
\end{aligned}
$$

Repeating the calculation until

$$
g_o\left(\boldsymbol{x}, t\right) = \chi_{\mathbb{A}}\left(\boldsymbol{x}\right)
$$

we recover (3.8).

References

[1] M. Avellaneda, *Ito processes, continuous-time martingales and Girsanov's Theorem*, lecture notes, `http://www.math.nyu.edu/faculty/avellane/rm3.ps`

[2] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, `http://math.berkeley.edu/~evans/`