1 Introduction

Beside [3] alternative reference for the Ito process are [1], [2] and Varadhan's lecture notes [4].

2 Martingales and Doob-Meyer decomposition

This section follows [2]. In the last lecture we identified a d-dimensional Ito process as the stochastic differential

$$d\boldsymbol{\xi}_t = \boldsymbol{b}_t\, dt + \boldsymbol{\sigma}_t[d\boldsymbol{w}_t]$$
$$\sigma^i_t[d\boldsymbol{w}_t] := \sum_{j=1}^{n} \sigma^{ij}_t\, dw^j_t \qquad \begin{array}{l} i = 1, \dots, d \\ j = 1, \dots, n \end{array}$$

for $\boldsymbol{b}_t$, $\boldsymbol{\sigma}$ *non-anticipative* with respect to the Wiener-process. Alternatively we can write

$$\boldsymbol{\xi}_t = \boldsymbol{\xi}_o + \int_0^t ds\, \boldsymbol{b}_s + \int_0^t \boldsymbol{\sigma}_s[d\boldsymbol{w}_s]$$

The right hand side comprises three terms.

- An initial value $\boldsymbol{\xi}_o$, eventually deterministic, for the process.
- A *bounded variation* component

  $$\boldsymbol{B}_t := \int_0^t ds\, \boldsymbol{b}_s$$

  This means that for any vector component i of $\boldsymbol{B}$ we have

  $$\lim_{|\mathsf{p}|\downarrow 0} \sum_k (t_{k+1} - t_k)|B^i_{t_{k+1}} - B^i_{t_k}| = \int_0^t dt\, |B^i| < \infty$$

  which implies the vanishing of the quadratic variation.
- A *martingale* component

  $$\boldsymbol{M}_t = \int_0^t \boldsymbol{\sigma}_t[d\boldsymbol{w}_t]$$

  From the properties of the Ito-integral we have in fact:

  i Conservation of the average: for any t

  $$\prec \int_0^t \boldsymbol{\sigma}_s[d\boldsymbol{w}_s] \succ = 0$$

  since $\boldsymbol{\sigma}$ is non-anticipating.

  ii finite quadratic variation

  $$\prec \int_0^t \sigma^i_s[d\boldsymbol{w}_s] \int_0^t \sigma^j_{s'}[d\boldsymbol{w}_{s'}] \succ = \int_0^t ds \sigma^{ik}_s \sigma^{jk}_s < 0$$

  As usual Einstein convention is implied: repeated indices stand for index contraction.

The representation of an Ito process in the form

$$\boldsymbol{\xi}_t = \boldsymbol{\xi}_o + \boldsymbol{B}_t + \boldsymbol{M}_t$$

is often referred to as the *Doob-Meyer representation*

Example 2.1 (*Exponential martingale*)**.** Let us consider the process

$$\xi_t = e^{\lambda\, w_t - \frac{\lambda^2\, t}{2}}\,\xi_o \tag{2.1}$$

by Ito lemma we have

$$d\xi_t = \lambda\, dw_t\, e^{\lambda\, w_t - \frac{\lambda^2\, t}{2}}\,\xi_o = \lambda\,\xi_t dw_t$$

If we recast the Ito differential into Doob-Meyer form we find

$$\xi_t = \xi_o + \lambda \int_0^t dw_s\,\xi_s$$

The exponential process does not have bounded variation component. It is therefore a *martingale*

$$\prec \xi_t \succ = \prec \xi_o \succ$$

if

$$\prec |\xi_o| \succ < \infty$$

3 Stochastic calculus with Hermite polynomials

This section expands example D.3 of chapter 4 of [3].

Proposition 3.1 (*Expansion of the transition probability of the Wiener process*)**.** *We have*

$$\frac{e^{-\frac{(x-y)^2}{2\,t}}}{\sqrt{2\,\pi\,t}} = \sum_{n=0}^{\infty} \frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} \left(\frac{y}{t}\right)^n h_n\,(x,t)$$

where

$$h_n\,(x\,,t) = \frac{(-t)^n}{\Gamma(n+1)}\, e^{\frac{x^2}{2\,t}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2\,t}} \tag{3.1}$$

Proof. The n-th order of the Taylor expansion can be couched into the form

$$\frac{y^n}{\Gamma(n+1)} \left.\frac{d^n}{dz^n}\right|_{z=0} \frac{e^{-\frac{(x-z)^2}{2\,t}}}{\sqrt{2\,\pi\,t}} := \frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} \left(\frac{y}{t}\right)^n h_n\,(x,t)$$

whence we can calculate the explicit form of the polynomial h_n. Namely

$$\frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} \left(\frac{y}{t}\right)^n h_n\,(x\,,t) = \\ \frac{y^n}{\Gamma(n+1)} \left.\frac{d^n}{dz^n}\right|_{z=0} \sum_{k=0}^{\infty} \int_{\mathbb{R}} e^{\imath\, p x - \frac{t\, p^2}{2}} \frac{(-\imath\, p\, z)^k}{\Gamma(k+1)} = \frac{y^n}{\Gamma(n+1)} \int_{\mathbb{R}} e^{\imath\, p x - \frac{t\, p^2}{2}} (-\imath\, p)^n$$

Observing that powers of p are generated by taking derivatives with respect to x we get into

$$\frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}}\left(\frac{y}{t}\right)^n h_n\,(x\,,t)=\frac{(-y)^n}{\Gamma(n+1)}\frac{d^n}{dx^n}\int_{\mathbb{R}} e^{\imath\,px-\frac{t\,p^2}{2}}$$

Performing the integral and contrasting the left to the right hand side yields the claim. □

The polynomials h_n defined by (3.1) are called the Hermite polynomials. It is readily checked that they enjoy the scaling property

$$h_n(\lambda\,x\,,\lambda^2\,t)=\lambda^n h_n(x\,,t)\qquad\Rightarrow\qquad(x\,\partial_x+2\,t\partial_t)\,h_n(x\,,t)=n\,h_n(x\,,t)$$

Furthermore

Proposition 3.2 (*Expected value of Hermite polynomials*)**.**

$$\prec h_n\,(w_t+x\,,t)\succ=\frac{x^n}{\Gamma(n+1)}=h_n(x,0) \tag{3.2}$$

Proof.

$$\prec h_n\,(w_t+x\,,t)\succ:=\int_{\mathbb{R}} dy\,h_n(y,t)\,\frac{e^{-\frac{(y-x)^2}{2\,t}}}{\sqrt{2\,\pi\,t}}=\frac{(-t)^n}{\Gamma(n+1)}\int_{\mathbb{R}} dy\,\frac{e^{-\frac{y^2}{2\,t}+\frac{x\,y}{t}}}{\sqrt{2\,\pi\,t}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2\,t}}$$

integrating by parts yields the claim. □

The reason why the expectation value is preserved is that the differential of Hermite along realizations of the Wiener process takes the form.

Proposition 3.3 (*Stochastic differential of Hermite polynomials*)**.**

$$dh_n(w_t,t)=dw_t\partial_{w_t}h_n(w_t,t)$$

Proof. By Ito lemma we have

$$dh_n(w_t,t)=dt\left(\partial_t+\frac{1}{2}\partial_{w_t}^2\right)\,h_n(w_t,t)+dw_t\partial_{w_t}h_n(w_t,t)$$

In order to prove the claim we need to show that

$$\left(\partial_t+\frac{1}{2}\partial_{w_t}^2\right)\,h_n(w_t,t)=0$$

Such result can be achieved by direct calculation. It is instructive to proceed in a slightly indirect way. For any $t>0$

$$\begin{aligned}0=&\left(\partial_t-\frac{1}{2}\partial_x^2\right)\frac{e^{-\frac{(x-y)^2}{2\,t}}}{\sqrt{2\,\pi\,t}}\\ &\sum_{n=0}^{\infty}\frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}}\left(\frac{y}{t}\right)^n\frac{1}{t}\left(-n\,+t\,\partial_t-\frac{t}{2}\partial_x^2+x\,\partial_x\right)h_n=-\sum_{n=0}^{\infty}\frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}}\left(\frac{y}{t}\right)^n\left(\partial_t+\frac{1}{2}\partial_x^2\right)\,h_n\end{aligned}$$

which implies

$$\left(\partial_t+\frac{1}{2}\partial_x^2\right)\,h_n=0$$

as each of these multiply positive definite terms of different order in y. □

We have therefore a probabilistic interpretation of the statistical conservation law

$$h_n(w_t + x, t) = h_n(x, 0) + \int_0^t dw_t \, \partial_{w_t} h_n(w_t, t)$$

From the property of the Ito integral

$$\prec h_n(w_t + x, t) \succ = h_n(x, 0) = \frac{x^n}{\Gamma(n+1)}$$

3.1 Recursion relation and multiple integrals over the Wiener process

Proposition 3.4 (*Integrals over Hermite polynomials*).

$$\int_0^t dw_s \, h_n(w_s, s) = h_{n+1}(w_s, s)$$

Proof. Consider the exponential martingale process (2.1). It satisfies

$$e^{\lambda w_t - \frac{\lambda^2 t}{2}} = 1 + \lambda \int_0^t dw_s \, e^{\lambda w_s - \frac{\lambda^2 s}{2}}$$

whence

$$\frac{d^n}{d\lambda^n}\bigg|_{\lambda=0} e^{\lambda w_t - \frac{\lambda^2 t}{2}} = \frac{d^n}{d\lambda^n}\bigg|_{\lambda=0} \lambda \int_0^t dw_s \, e^{\lambda w_s - \frac{\lambda^2 s}{2}}$$

Contrasting the left-hand side with the definition of Hermite polynomials we conclude

$$\frac{d^n}{d\lambda^n}\bigg|_{\lambda=0} e^{\lambda w_t - \frac{\lambda^2 t}{2}} = t^n \, \frac{d^n}{dz^n}\bigg|_{z=0} e^{\frac{z}{t} w_t - \frac{z^2}{2t}} = \Gamma(n+1) \, h_n(w_t \, , t)$$

The right hand side is

$$\frac{d^n}{d\lambda^n}\bigg|_{\lambda=0} \lambda \int_0^t dw_s \, e^{\lambda w_s - \frac{\lambda^2 s}{2}} = n \, \frac{d^{n-1}}{d\lambda^{n-1}}\bigg|_{\lambda=0} \int_0^t dw_s \, e^{\lambda w_s - \frac{\lambda^2 s}{2}} = \Gamma(n+1) \int_0^t dw_s \, h_{n-1}(w_s \, , s)$$

We have therefore proved that

$$h_n(w_t \, , t) = \int_0^t dw_s \, h_{n-1}(w_s \, , s)$$

□

An important consequence is the following. Since

$$h_0(w_t, t) = 1$$

we have that

$$\int_0^t dw_s = \int_0^t dw_s \, h_0(w_s \, , s) = h_1(w_t, t)$$

and

$$\int_0^t dw_{s_1} \int_0^{s_1} dw_{s_0} = h_2(w_t, t)$$

or in full generality

$$\int_0^t dw_{s_1} \prod_{i=1}^{n-1} \int_0^{s_{i-1}} dw_{s_{i-1}} = h_n(w_t, t)$$

4 The Stratonovich integral

This section follows section E of chapter 6 of [3]. We have seen that for

$$\theta_k = s\,t_k + (1-s)\,t_{k-1} \qquad \forall\, s \,\in\, [0,1] \tag{4.1}$$

the sum

$$\sum_{k=1}^{n} w_{\theta_k}\,(w_{t_k} - w_{t_{k-1}}) = \frac{w_{t_n}^2}{2} - \sum_k \left[\frac{(w_{t_k} - w_{\theta_k})^2}{2} - \frac{(w_{\theta_k} - w_{t_{k-1}})^2}{2}\right]$$

in $\mathbb{L}^2(\Omega)$ converges to

$$\int_0^t w_s^{(\theta)}\,dw_s = \frac{w_t^2}{2} - \frac{t\,(1-2\,s)}{2}$$

Choosing $s = 1/2$ the second term on the right hand side disappears and we recover the result from ordinary calculus. The example suggests to define

Fisk-Stratonovich integral	$\int_0^t dw_s \diamond \xi_s := \lim_{\lvert \mathsf{p}_{(n)} \rvert \downarrow 0} \sum_{t_k \in \mathsf{p}_n} \xi_{\frac{t_{k-1}+t_k}{2}}\,(w_{t_k} - w_{t_{k-1}})$	(4.2)

where $\{\mathsf{p}_n\}_{n=0}^{\infty}$ is a sequence of refining partitions of $[0,t]$. Note that

$$\begin{aligned} &\xi_{\frac{t_{k+1}+t_k}{2}} - \frac{\xi_{t_{k+1}} + \xi_{t_k}}{2} = \\ &\quad \frac{\xi_{t_{k+1} - \frac{t_{k+1}-t_k}{2}} - \xi_{t_{k+1}}}{2} + \frac{\xi_{t_k + \frac{t_{k+1}-t_k}{2}} - \xi_{t_k}}{2} = O(\xi_{t_{k+1}} - \xi_{t_k})^2 \end{aligned}$$

Thus we can equivalently write

$$\int_0^t dw_s \diamond \xi_s := \lim_{\lvert \mathsf{p}_{(n)} \rvert \downarrow 0} \sum_{t_k \in \mathsf{p}_n} \frac{\xi_{t_{k+1}} + \xi_{t_k}}{2}\,(w_{t_k} - w_{t_{k-1}})$$

As in the Ito case the limit converges in mean square sense. At variance with the Ito case, the integrand in the definition (4.2) is anticipating:

$$\prec \xi_t \diamond dw_t \succ \neq \prec \xi_t \succ \prec dw_t \succ = 0$$

Thus the martingale property of the Ito integral is lost. To appreciate the advantage of the definition consider

$$\int_0^t dw_s \diamond w_s = \lim_{\lvert \mathsf{p}_{(n)} \rvert \downarrow 0} \sum_{t_k \in \mathsf{p}_n} \frac{(w_{t_k} + w_{t_{k-1}})(w_{t_k} - w_{t_{k-1}})}{2} = \frac{w_t^2}{2} \tag{4.3}$$

in agreement with the rules of *ordinary differential calculus*. The example illustrates the general situation.

4.1 Relation with the Ito differential

Let us consider

$$\xi_t = g(\chi_t, t) \tag{4.4}$$

with

$$d\chi_t = b(\chi_t, t)\, dt + \sigma(\chi_t, t)\, dw_t \tag{4.5}$$

then by Ito lemma we can write

$$d\xi_t = dg(\chi_t, t) = dt \left\{ \partial_t + b_t\, \partial_{\chi_t} + \frac{\sigma^2}{2} \partial^2_{\chi_t} \right\} g + dw_t\, \sigma_t\, \partial_{\chi_t} g \tag{4.6}$$

and use the this result to establish the relation between the Fisk-Stratonovich and the Ito integral. Namely given a non-anticipating process η_t we can couch the definition of the Fisk-Stratonovich integral into the form

$$\int_0^t dw_s \diamond \eta_s = \lim_{|\mathsf{p}_{(n)}| \downarrow 0} \sum_{t_k \in \mathsf{p}_n} \left\{ \eta_{t_{k-1}} (w_{t_k} - w_{t_{k-1}}) + \frac{(\eta_{t_{k-1}} - \eta_{t_k})(w_{t_k} - w_{t_{k-1}})}{2} \right\}$$

In the literature the latter equality is sometimes written in the continuum limit as

$$\int_0^t dw_s \diamond \eta_s = \int_0^t dw_s\, \eta_s + \langle \eta\, , w \rangle_t$$

where $\langle \xi\, , w \rangle_t$ is *quadratic co-variation* of the processes ξ_t and w_t. The essential point is that in the limit (which converges in the mean square sense under our hypotheses) the quadratic co-variation receives finite contributions only from the term proportional to the increment of the Wiener process

$$dw_t \sim O(\sqrt{dt}) \qquad \Rightarrow \qquad dw_t^2 \sim O(dt)$$

In such a case if

$$\eta_t = f(\chi_t, t)$$

we find

Stratonovich to Ito conversion

$$\int_0^t dw_s \diamond f(\chi_s, s) = \int_0^t dw_s\, f(\chi_s, s) + \frac{1}{2} \int_0^t ds\, \sigma(\chi_s, s)\, \partial_{\chi_s} f(\chi_s, s) \tag{4.7}$$

In particular for

$$f(\chi_t, t) = \sigma(\chi_t, t) \partial_{\chi_t} g(\chi_t, t)$$

we obtain

$$\begin{aligned} dw_t \diamond [\sigma(\chi_t, t) \partial_{\chi_t} g(\chi_t, t)] = \\ dw_t\, \sigma(\chi_t, t) \partial_{\chi_t} g(\chi_t, t) + \frac{dt}{2}\, \sigma(\chi_t, t)\, \partial_{\chi_t} [\sigma(\chi_t, t)\, \partial_{\chi_t} g(\chi_t, t)] \end{aligned}$$

which allows us to write

$$d\xi_t = dg(\chi_t, t) = dt \left\{ \partial_t + \left[b - \frac{\sigma}{2} (\partial_{\chi_t} \sigma) \right] \partial_{\chi_t} \right\} g + dw_t \diamond [\sigma \partial_{\chi_t} g] \tag{4.8}$$

As expected, the right hand side does not include any-longer a second derivative of g, the hallmark of Ito lemma. The function g is, however, transported by the Stratonovich stochastic differential

$$d\xi_t = dt \left[b_t - \frac{1}{2} (\sigma \partial_{\chi_t} \sigma)_t \right] + dw_t \sigma_t$$

4.2 Examples

- Consider the process

$$\xi_t = w_t^2$$

In such a case the role of the process χ_t of the previous section is played by the Brownian motion itself

$$\chi_t = w_t \qquad \Rightarrow \qquad d\chi_t = dw_t$$

i.e. $b = 0$ and $\sigma = 1$ in (4.5). Ito lemma yields

$$d\xi_t = dg(w_t) = 2\, w_t\, dw_t + dt$$

The differential admits the equivalent Stratonovich representation

$$d\xi_t = 2\, w_t \diamond dw_t$$

with again $\chi_t = w_t$.

- Consider now

$$\xi_t = \chi_t^2$$

with

$$d\chi_t = \chi_t\, dt + \chi_t\, dw_t \tag{4.9}$$

This case corresponds to

$$b = \sigma = \chi_t$$

in (4.5). It follows by Ito lemma

$$d\xi_t = 3\chi_t^2 dt + 2\, \chi_t^2\, dw_t$$

On the other hand (4.9) admits the Stratonovich representation

$$d\chi_t = \frac{\chi_t}{2}\, dt + \chi_t \diamond dw_t$$

whence

$$d\xi_t = \chi_t^2 dt + 2\, \chi_t^2 \diamond dw_t$$

References

[1] M. Avellaneda, *http://www.math.nyu.edu/faculty/avellane/rm2.ps*, Mathematics of Finance II lecture notes, `http://www.math.nyu.edu/faculty/avellane/rm2.ps` 1

[2] M. Avellaneda, *Ito processes, continuous-time martingales and Girsanov's Theorem*, lecture notes, `http://www.math.nyu.edu/faculty/avellane/rm3.ps` 1

[3] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, `http://math.berkeley.edu/~evans/` 1, 2, 5

[4] S.R.S. Varadhan, *Stochastic Calculus, lecture notes*, `http://www.math.nyu.edu/faculty/varadhan/stochastic.fall08.html` 1