1 Introduction

The content of these notes is also covered by chapter 4 section A, B and C of [1]. An alternative reference for Brownian motion and it properties is provided by Varadhan's lecture notes [2].

2 Partitions

Definition 2.1 (*Partition*)**.** *If $I = [x_-, x_+] \subset \mathbb{R}$ is an interval a partition p (subdivision) of I is a finite sequence $\{x_k\}_{k=1}^n$ of points in I such that*

$$x_- = x_1 < \ldots < x_n = x_+$$

Definition 2.2 (*Mesh of a partition*)**.** *The mesh size of a partition p of an interval $I = [x_-, x_+]$ is*

$$|\mathsf{p}| = \max_{1 \leq k \leq n} |x_{k+1} - x_k|$$

Definition 2.3 (*Refinement of a partition*)**.** *The refinement of a partition p of the interval I is another partition p' that contains all the points from p and* some additional points*, again sorted by order of magnitude.*

3 Quadratic variation

Definition 3.1 (*Quadratic (co-)variation*)**.** *Let $\{\xi_t \,|\, t \geq 0\}$ and $\{\chi_t \,|\, t \geq 0\}$ two stochastic processes. The limit (in mean square sense)*

$$\langle \xi, \chi \rangle_t = \lim_{|\mathsf{p}_{(n)}| \downarrow 0} \sum_{t_k \in \mathsf{p}_n} (\xi_{t_k} - \xi_{t_{k-1}})(\chi_{t_k} - \chi_{t_{k-1}})$$

is called the quadratic co-variation of the processes. In particular

$$\langle \xi, \xi \rangle_t = \lim_{|\mathsf{p}_{(n)}| \downarrow 0} \sum_{t_k \in \mathsf{p}_n} (\xi_{t_k} - \xi_{t_{k-1}})^2$$

is called the quadratic variation of ξ_t.

For the Brownian motion we have

Proposition 3.1 (*Quadratic variation of the B.M.*)**.** *The quadratic variation of the Brownian motion*

$$\langle w, w \rangle_t = \sigma^2 t$$

in the sense of $\mathbb{L}^2(\Omega)$.

Proof. Set

$$Q_n := \sum_{k=0}^{m_n - 1} (w_{t_k} - w_{t_{k-1}})^2$$

we have then

$$\prec [Q_n - (t_b - t_a)]^2 \succ = \sum_{k\,l=1}^{m_n-1} \prec \left[(w_{t_k} - w_{t_{k-1}})^2 - \sigma^2 (t_k - t_{k-1})\right] \left[(w_{t_l} - w_{t_{l-1}})^2 - \sigma^2 (t_l - t_{l-1})\right] \succ$$

For non-overalapping intervals, the averaged quantities are independent random variables with zero average. The only contributions to the sum come from overlapping intervals:

$$\prec [Q_n - \sigma^2 (t_b - t_a)]^2 \succ = \sum_{k=1}^{m_n - 1} \prec \left[(w_{t_k} - w_{t_{k-1}})^2 - (t_k - t_{k-1})\right]^2 \succ = 2\,\sigma^4 \sum_{k=1}^{m_n - 1} (t_k - t_{k-1})^2$$

whence

$$\prec [Q_n - (t_b - t_a)]^2 \succ \leq 2\,\sigma^4 \,(t_b - t_a) \max_k (t_k - t_{k-1}) \overset{\max_k (t_k - t_{k-1}) \to 0}{\rightarrow} 0$$

□

Some observations are in order

- The finite value of the quadratic variation motivates the estimate

  $$dw_t \sim O(\sqrt{dt})$$

  for typical increments of the Wiener process.

- The finite value of the quadratic variation is a further manifestation of the non-differentiability of the Wiener process. Namely, the *intermediate value theorem* implies for a differentiable function

  $$\lim_{dt \downarrow 0} \sum_k |f(t_k) - f(t_{k-1})| \approx \int_0^t dt\, |f'(t)|$$

  Irrespective of differentiability

  $$\sum_k [f(t_k) - f(t_{k-1})]^2 \leq \max_k |f(t_k) - f(t_{k-1})| \sum_k |f(t_k) - f(t_{k-1})|$$

  so that for a differentiable function

  $$\sum_k [f(t_k) - f(t_{k-1})]^2 \leq \max_k |f(t_k) - f(t_{k-1})| \int_0^t dt\, |f'(t)| \overset{\max_k (t_k - t_{k-1}) \to 0}{\rightarrow} 0$$

  In the case of the Wiener process the finiteness of the right hand side implies

  $$\sum_k |f(t_k) - f(t_{k-1})| \overset{\max_k (t_k - t_{k-1}) \to 0}{\uparrow} \infty$$

4 Stochastic integrals

Let f an analytic function

$$f\colon\, \mathbb{R} \to \mathbb{R}$$

we would like to make sense of the functional of the Wiener process

$$I = \int_0^t f(w_s)\, dw_s \qquad \text{mathematics notation} \tag{4.1}$$

The *formal* relation between Wiener process and white noise, is often sometime used to write the above integral as

$$I = \int_0^t f(w_s)\, \eta_s\, ds \qquad \text{physics notation}$$

It is important to realise that the integral **cannot be interpreted as a Lebesgue-Stieltjes integral**. Namely take

$$f(w_s) = w_s$$

and suppose to define the integral as

$$\int_0^t w_s\, dw_s = \lim_{n\uparrow\infty} \sum_{k=1}^{n} w_{\theta_k}\,(w_{t_k} - w_{t_{k-1}}) \tag{4.2}$$

As n increases the $\{t_k\}_{k=1}^n$ describe sequences of refining partitions (see appendix 2) of the interval $[0,t]$. The point θ_k is chosen arbitrarily in $[t_{k-1}\,, t_k]$:

$$\theta_k = s\, t_k + (1-s)\, t_{k-1} \qquad \forall\, s\, \in\, [0,1] \tag{4.3}$$

A necessary condition for the interpretation of (4.2) as a Lebesgue-Stieltjes integral to hold true is that the right hand side must be independent of way θ_k is sampled. Instead, we find that even the average of the integral does not satisfy the requirement:

$$\begin{aligned} \prec \int_0^t w_s\, dw_s \succ &= \lim_{|\mathsf{p}|\downarrow 0} \sum_{k=1}^{n} \{\prec w_{\theta_k}\, w_{t_k} \succ - \prec w_{\theta_k}\, w_{t_{k-1}} \succ\} \\ &= \lim_{n\uparrow\infty} \sum_{k=1}^{n} \sigma^2\,(\theta_k - t_{k-1}) = \lim_{n\uparrow\infty} \sum_{k=1}^{n} \sigma^2\, s(t_k - t_{k-1}) = \sigma^2\, s \end{aligned}$$

4.1 Filtration

Definition 4.1 (*Filtration*)**.** *A family $\{\mathcal{F}_t\,:\, t \geq 0\}$ of σ-algebras is called* **non-anticipating** *with respect to the family of σ-algebras $\{\mathcal{W}_t\,:\, t \geq 0\}$ induced by a Wiener process $\{w_t\,:\, t \geq 0\}$ if it satisfies*

- $\mathcal{F}_t \subseteq \mathcal{F}_{t'}$ *for all* $t\, \leq\, t'$
- $\mathcal{W}_t \subseteq \mathcal{F}_t$ *for all* $t\, \geq\, 0$
- $\mathcal{F}_t$ *is independent of* $\{\mathcal{W}_{t'} - \mathcal{W}_t\,:\, t \leq t'\}$ *for all* $t\, \leq\, t'$.

A non-anicipating family of σ-algebras is also referred to as **filtration**.

Example 4.1 (*Non-anticipative vs anticipative*)**.** Let w_t a Wiener process for all $t \geq\, 0$, the function

$$f(t) = \begin{cases} 0 \quad \text{if} & \max\limits_{0\leq s\leq t} w_s \leq 1 \\ 1 \quad \text{if} & \max\limits_{0\leq s\leq t} \; > 1 \end{cases}$$

is *non-anticpative* as it depends on the Wiener process up to the time t when the function is evaluated. On the other hand for any $T > t$ the function

$$g(t) = \begin{cases} 0 \quad \text{if} & \max\limits_{0\leq s\leq T} w_s \leq 1 \\ 1 \quad \text{if} & \max\limits_{0\leq s\leq T} w_s \; > 1 \end{cases}$$

is *anticpative* as it depends on realizations of the Wiener process for times s posterior to the sampling time t.

4.2 Ito integral

Let suppose that ξ_t is a stochastic process satisfying the properties

ii Non anticipating: ξ_t may depend only on w_s with $s \leq t$. As a consequence ξ_t and dw_t are **independent variables**

$$\prec \xi_t dw_t \succ = \prec \xi_t \succ \prec dw_t \succ = 0$$

ii mean square integrability:

$$\prec \int_0^t ds\, \xi_s^2 \succ < \infty$$

For any stochastic process ξ_t satisfying the above two properties we can define the Ito integral

$$\text{Ito prescription} \qquad \int_0^t dw_s\, \xi_s := \lim_{|\mathsf{p}_{(n)}|\downarrow 0} \sum_{k=1}^{n} \xi_{t_{k-1}} (w_{t_k} - w_{t_{k-1}}) \tag{4.4}$$

where $\{\mathsf{p}_n\}_{n=1}^{\infty}$ is a sequence of refining partitions $\mathsf{p}_n = \{t_k\}_{k=0}^{n}$ of the interval $[0, t]$ with mesh size $|\mathsf{p}_n|$ (see appendix 2). Note that the approximating sums

$$I_n = \sum_{k=1}^{n} \xi_{t_{k-1}} (w_{t_k} - w_{t_{k-1}})$$

are defined in the Ito prescription by setting s to zero in (4.3). The convergence of (4.4) has to be understood in the mean square sense i.e.

$$\prec (I_n - I_m)^2 \succ \stackrel{n,m\uparrow\infty}{\rightarrow} 0$$

The hypotheses i, ii (4.4) assumed in the definition imply that

i Ito integrals have zero average

$$\prec \int_0^t dw_s\, \xi_s \succ = 0 \tag{4.5}$$

ii the *mean square integrability* property

$$\begin{aligned} \prec \left(\int_0^t dw_s\, \xi_s \right)^2 \succ &= \prec \int_0^t \int_0^t dw_s dw_{s'}\, \xi_s \xi_{s'} \succ \\ &= \int_0^t \int_0^t \prec dw_s dw_{s'} \succ \prec \xi_s \xi_{s'} \succ = \int_0^t \int_0^t ds ds'\, \delta(s - s') \prec \xi_s \xi_{s'} \succ = \int_0^t ds \prec \xi_s^2 \succ \end{aligned} \tag{4.6}$$

using the formal relation

$$dw_t = \eta_t\, dt$$

between white noise and Wiener increment. The proof in $\mathbb{L}^2(\Omega)$-sense of the chain of equalities in (4.6) proceeds by considering finite approximants to the integral as in the case of the proof of finiteness of the quadratic variation of the Wiener process. The same method will be outlined below in relation to the proof of Ito lemma.

In particular for

$$\sum_{k=1}^{n} w_{t_{k-1}}(w_{t_k} - w_{t_{k-1}}) = \sum_{k=1}^{n} \frac{w_{t_k}^2 - w_{t_{k-1}}^2}{2} - \sum_{k=1}^{n} \frac{(w_{t_k} - w_{t_{k-1}})^2}{2}$$

upon applying the quadratic variation lemma we find

$$\int_0^t dw_s\, w_s = \frac{w_t^2}{2} - \frac{t}{2}$$

5 (Döblin-Gih'man-) Ito lemma

Proposition 5.1 (*Functionals of the Wiener process*). *Let*

$$F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$$

a smooth function with bounded derivatives in all its arguments. Let $\boldsymbol{w}_t$ be a d-dimensional Wiener process with diffusion coefficient σ on each component. Then

$$F(\boldsymbol{w}_t, t) = F(\boldsymbol{0}, 0) + \int_0^t d\boldsymbol{w}_s \cdot \partial_{\boldsymbol{w}_s} F(\boldsymbol{w}_s, s) + \int_0^t ds \left\{ \partial_s F(\boldsymbol{w}_s, s) + \frac{\sigma^2}{2} \partial_{\boldsymbol{w}_s}^2 F(\boldsymbol{w}_s, s) \right\}$$

Sketch of the proof. Consider the Taylor expansion of F around a point $(\boldsymbol{w}_{\bar{t}}, \bar{t})$. We have

$$\begin{aligned} F(\boldsymbol{w}_{\bar{t}+h}, \bar{t} + h) - F(\boldsymbol{w}_{\bar{t}}, \bar{t}) &= h\, \partial_{\bar{t}} F(\boldsymbol{w}_{\bar{t}}, \bar{t}) + (\boldsymbol{w}_{\bar{t}+h} - \boldsymbol{w}_{\bar{t}}) \cdot \partial_{\boldsymbol{w}_{\bar{t}}} F(\boldsymbol{w}_{\bar{t}}, \bar{t}) \\ &+ \frac{1}{2}(w_{\bar{t}+h} - w_{\bar{t}})^i (w_{\bar{t}+h} - w_{\bar{t}})^j \partial_{w_{\bar{t}}^i} \partial_{w_{\bar{t}}^j} F(\boldsymbol{w}_{\bar{t}}, \bar{t}) + h\, (\boldsymbol{w}_{\bar{t}+h} - \boldsymbol{w}_{\bar{t}}) \cdot \partial_{\boldsymbol{w}_{\bar{t}}} \partial_{\bar{t}}\, F(\boldsymbol{w}_{\bar{t}}, \bar{t}) \\ &+ \frac{h^2}{2} \partial_{\bar{t}}^2\, F(\boldsymbol{w}_{\bar{t}}, \bar{t}) + \frac{1}{6}(w_{\bar{t}+h} - w_{\bar{t}})^i (w_{\bar{t}+h} - w_{\bar{t}})^j (w_{\bar{t}+h} - w_{\bar{t}})^k \partial_{w_{\bar{t}}^i} \partial_{w_{\bar{t}}^j} \partial_{w_{\bar{t}}^k} F(\boldsymbol{w}_{\bar{t}}, \bar{t}) + \dots \end{aligned}$$

having adopted Einstein convention on repeated indices. We can use Taylor expansions like the above to evaluate increments of F on the elements of any partition of $[0, t]$. By subsequently refining partitions we can in such a way Adding up the increments we observe that

$$\begin{aligned} &\sum_k \left\{ (t_{k+1} - t_k)\, \partial_{t_k} F(\boldsymbol{w}_{t_k}, t_k) + (\boldsymbol{w}_{t_{k+1}} - \boldsymbol{w}_{t_k}) \cdot \partial_{\boldsymbol{w}_{t_k}} F(\boldsymbol{w}_{t_k}, t_k) \right\} \\ &\overset{|\mathsf{p}| \downarrow 0}{\to} \int_0^t ds\, \partial_s F(\boldsymbol{w}_s, s) + \int_0^t d\boldsymbol{w}_s \cdot \partial_{\boldsymbol{w}_s} F(\boldsymbol{w}_s, s) \end{aligned}$$

for p the mesh of the partition. We need to show that in the same $\mathbb{L}^2\,(\Omega)$ sense in which we defined the Ito integral a third contribution. Since for some $C_p > 0$

$$\prec ||\boldsymbol{w}_{t+h} - \boldsymbol{w}_t||^p\, h^q \succ = C_p\, h^{\frac{p}{2} + q}$$

The only terms which do not vanish asymptotically as the partition mesh tends to zero are those corresponding to

$$(p\,, q) = (2\,, 0)$$

We need therefore to study in mean square sense

$$Q_n = \sum_{k=0}^{m_n} [(w_{t_{k+1}} - w_{t_k})^i (w_{t_{k+1}} - w_{t_k})^j - \delta^{ij} \sigma^2\, (t_{k+1} - t_k)] \partial_{w_{t_k}^i} \partial_{w_{t_k}^j} F(\boldsymbol{w}_{t_k}, t_k)$$

we have

- mean value: using the independence of the increments

$$\prec Q_n \succ = 0$$

- mean square value:

$$\prec Q_n^2 \succ = \sum_{k,l=0}^{m_n} \prec \Delta_k^{ij} \Delta_l^{i'j'} [\partial_{w^i_{t_k}} \partial_{w^j_{t_k}} F(\boldsymbol{w}_{t_k}, t_k)][\partial_{w^{i'}_{t_l}} \partial_{w^{j'}_{t_l}} F(\boldsymbol{w}_{t_l}, t_l)] \succ$$

with

$$\Delta_k^{ij} := (w_{t_{k+1}} - w_{t_k})^i (w_{t_{k+1}} - w_{t_k})^j - \delta^{ij} \sigma^2 (t_{k+1} - t_k)$$

As for the quadratic variation of the Wiener process, the Δ_k^{ij} for non overlapping intervals are independent variables with zero mean:

$$\begin{aligned}
\prec Q_n^2 \succ &= \sum_{k=0}^{m_n} \prec \Delta_k^{ij} \Delta_k^{i'j'} [\partial_{w^i_{t_k}} \partial_{w^j_{t_k}} F(\boldsymbol{w}_{t_k}, t_k)][\partial_{w^{i'}_{t_k}} \partial_{w^{j'}_{t_k}} F(\boldsymbol{w}_{t_k}, t_k)] \succ \\
&= \sigma^4 \sum_{k=0}^{m_n} (t_{k+1} - t_k)^2 (\delta^{ij'} \delta^{i'j} + \delta^{i'i} \delta^{jj'}) \prec [\partial_{w^i_{t_k}} \partial_{w^j_{t_k}} F(\boldsymbol{w}_{t_k}, t_k)][\partial_{w^{i'}_{t_k}} \partial_{w^{j'}_{t_k}} F(\boldsymbol{w}_{t_k}, t_k)] \succ \\
&\leq 2\,\sigma^4 |\mathsf{p}| \int_0^t ds \prec [\partial_{w^i_{t_k}} \partial_{w^j_{t_k}} F(\boldsymbol{w}_{t_k}, t_k)][\partial_{w^i_{t_k}} \partial_{w^j_{t_k}} F(\boldsymbol{w}_{t_k}, t_k)] \succ \overset{|\mathsf{p}|\downarrow 0}{\to} 0
\end{aligned}$$

We have so heuristically shown that in mean square sense

$$\sum_{k=0}^{m_n} (w_{t_{k+1}} - w_{t_k})^i (w_{t_{k+1}} - w_{t_k})^j \partial_{w^i_{t_k}} \partial_{w^j_{t_k}} F(\boldsymbol{w}_{t_k}, t_k) \overset{|\mathsf{p}|\downarrow 0}{\to} \sigma^2 \int_0^t ds\, \partial^2_{\boldsymbol{w}_s} F(\boldsymbol{w}_s, s)$$

and thus completed the sketch of the proof. □

6 (Döblin-Gih'man-) Ito calculus

Definition 6.1 (*Ito process*)**.** *We say that ξ_t for $t \geq 0$ is an Ito process if there exist a Wiener motion measure and two non-anticipative functions A_t, B_t for all $t \geq 0$ such that*

$$\xi_t = \xi_o + \int_0^t ds\, A_s + \int_0^t dw_s\, B_s$$

The generalization to the d-dimensional case is straightforward. Take two non-anticipative $\boldsymbol{A}_t$ and $\boldsymbol{B}_t$ respectively $\mathbb{R}^d$ and $\mathbb{R}^{d' \times d}$ valued fields

$$\boldsymbol{\xi}_t = \boldsymbol{\xi}_o + \int_0^t ds\, \boldsymbol{A}_s + \int_0^t d\boldsymbol{w}_s \cdot \boldsymbol{B}_s$$

where now

$$(d\boldsymbol{w}_s \cdot \boldsymbol{B}_s)^i := \sum_{j=1}^{d'} B_s^{ij} dw_s^j$$

Contrasting the definition with the content of Ito lemma, an immediate consequence is that

$$F^l(\boldsymbol{w}_t, t) = F^l(\mathbf{0}, 0) + \int_0^t ds\, A_s^l + \int_0^t dw_s^k\, B_s^{lk}$$

requires

$$\boldsymbol{A}_t = \partial_t \boldsymbol{F}(\boldsymbol{w}_t, t) + \frac{\sigma^2}{2} \partial^2_{\boldsymbol{w}_t} \boldsymbol{F}(\boldsymbol{w}_t, t)$$

and

$$B_t^{ij} = \partial_{w_t^j} F^i(\boldsymbol{w}_t, t)$$

The pair $(\boldsymbol{A}_t\,, \boldsymbol{B}_t)$ is referred to as the *local coefficients* of the Ito process.

Proposition 6.1 (*Generalised (Döblin-Gih'man-)Ito lemma*)**.** *Let* $\boldsymbol{\xi}_t$ *a* d*-dimensiona Ito process with local coefficients* $(\boldsymbol{A}_t\,, \boldsymbol{B}_t)$*. Let* $\boldsymbol{F}$ *a smooth vector field*

$$\boldsymbol{F} : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{\tilde{d}}$$

with bounded derivatives. Then

$$\boldsymbol{\eta}_t = \boldsymbol{F}\,(\boldsymbol{\xi}_t, t)$$

is a $\tilde{d}$*-dimensional Ito process with local parameters*

$$\tilde{A}_t^l = \partial_t F^l + A^i \partial_{\xi_t^i} F^l + \frac{1}{2} B^{ik} B^{jk} \partial_{\xi_t^i} \partial_{\xi_t^j} F^l$$

and

$$\tilde{B}_t^{lk} = B^{ik} \partial_{\xi_t^i} F^l$$

References

[1] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, `http://math.berkeley.edu/~evans/` 1

[2] S.R.S. Varadhan, *Stochastic Calculus, lecture notes*, `http://www.math.nyu.edu/faculty/varadhan/stochastic.fall08.html` 1