1 Introduction

The content of these notes is also covered by chapter 3 section C of [1].

2 Properties of the Brownian motion

- Covariance

$$\prec w_t \, w_s \succ = \sigma^2 \, (t \wedge s) := R(t,s) \tag{2.1}$$

- Stationary increments: for any $t, s \in \mathbb{R}_+$

$$\prec (w_t - w_s)^2 \succ = \sigma^2 \, |t-s|$$

- Characteristic function: for $t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_m$

$$\prec e^{\imath \sum_{k=1}^{m-1} \lambda_k (w_{t_{k+1}} - w_{t_k})} \succ = \prod_{k=1}^{m-1} \prec e^{\imath \lambda_k (w_{t_{k+1}} - w_{t_k})} \succ = \prod_{k=1}^{m-1} e^{-\frac{\lambda_k^2 \sigma^2}{2}(t_{k+1} - t_k)}$$

3 Karhunen-Loève representation of the Wiener process

The Karhunen-Loève representation provides us with a convenient *deterministic* functional orthonormal basis $\{\psi_n\}_{n=1}^{\infty}$, which will allow us to write the Brownian motion

$$w_t : \Omega \times [0,T] \to \mathbb{R}$$

as a series

$$w_t = \sum_{n=0}^{\infty} c_n \psi_n(t)$$

Randomness is encoded in the coefficients

$$c_n = \int_0^T \frac{dt}{T} \, \psi_n(t) w_t$$

which need to form a sequence $\{c_n\}_{n=0}^{\infty}$ of independent Gaussian random variable with zero average and variance determined by the Karhunen-Loève representation.

Let us consider the space of the Lebesgue square integrable $\mathbb{L}^2([0,T])$ real functions on $[0,T]$. This space is an Hilbert space with respect to the scalar product:

$$< f, g > := \int_0^T \frac{dt}{T} \, f(t) g(t) \qquad f, g \in \mathbb{L}^2([0,T])$$

Proposition 3.1. *The covariance R of the Brownian motion (2.1) defines the* kernel *of an operator mapping $\mathbb{L}^2([0,T])$ into itself:*

$$g(t) = \int_0^T \frac{dt}{T} \, R(t\,,s) \, f(s) := R[f](t)$$

i.e.

$$f \in \mathbb{L}^2([0,T]) \to g \equiv R[f] \in \mathbb{L}^2([0,T])$$

Proof. The following chain of equalities holds by definition

$$||g||_2^2 :=< R[f]\,, R[f] >= \int_0^T \frac{dt}{T} \left[\int_0^T \frac{ds}{T}\, R(t,s)\, f(s) \right] \left[\int_0^T \frac{ds'}{T}\, R(t,s')\, f(s') \right]$$

where

$$\int_0^T \frac{ds}{T}\, R(t,s)\, f(s) = \sigma^2 \int_0^t \frac{ds}{T}\, s\, f(s) + \sigma^2\, t \int_t^T \frac{ds}{T}\, f(s)$$

so that by Lyapunov inequality

$$|R[f](t)| \leq \sigma^2\, T \int_0^T \frac{ds}{T} |f(s)| \;\leq\; \sigma^2\, T \left(\int_0^T \frac{ds}{T} |f(s)|^2 \right)^{1/2}$$

We conclude

$$||g||_2^2 \leq \sigma^4\, T^2 ||f||_2^2$$

□

The as an operator kernel $R(t,s)$ enjoys two further properties

- $R(t,s)$ is positive definite: i.e. for any collection of $\{c_i\}_{i=1}^n \in \mathbb{C}$ and sampling of the arguments $\{t_i\}_{i=1}^n$ we have

$$\sum_{ij=1}^n c_i R(t_i, s_i) c_j^* \geq\; 0$$

  Namely

$$\sum_{ij=1}^n c_i R(t_i, t_j) c_j^* = \sum_{i,j=1}^n \prec w_{t_i} w_{t_j} \succ c_i\, c_j^* =\prec \left| \sum_{i=1}^n c_i w_{t_i} \right|^2 \succ\geq 0$$

- The kernel is symmetric:

$$R(t,s) = R(s,t)$$

These properties of the kernel allows us to invoke

Theorem 3.1 (*Mercer*)**.** *Let R be a continuous symmetric non-negative definite kernel. Then there is an orthonormal basis $\{\psi_n\}_{n=0}^\infty$ of $\mathbb{L}^2([0,T])$ consisting of eigenfunctions of of the operator*

$$R[f] = \int_0^T \frac{ds}{T} R(t,s)\, f(s)$$

such that the corresponding sequence of eigenvalues $\{r_n\}_{n=0}^\infty$ is non-negative. The eigenfunctions corresponding to non-zero eigenvalues are continuous on $[0,T]$ and R admit the representation

$$R(t,s) = \sum_{n=0}^\infty r_n\, \psi_n(t)\, \psi_n(s)$$

Proof. The theorem is proved at pag. 138 of [2]. It can also be regarded as a consequence of the Hilbert-Schmidt theorem [3] □

In other words the operator specified by R is diagonalizable with discrete spectrum. The idea underlying the Karhunen-Loève representation is to use the orthonormal basis of eigenvectors to write the Brownian motion as the $\mathbb{L}^2$-convergent series:

$$w_t = \sum_{n=0}^{\infty} c_n \, \psi_n(t) \tag{3.1}$$

Namely, if the $\{c_n\}_{n=0}^{\infty}$ are Gaussian independent random variables satisfying

$$\prec c_n \succ = 0 \qquad \& \qquad \prec c_n^2 \succ = r_n$$

it follows immediately that (3.1) satisfies all the requirements to be a Brownian motion:

i Covariance:

$$\prec w_t \, w_s \succ = \sum_{n=0}^{\infty} r_n \, \psi_n(t)\psi_n(s) = R(t,s)$$

ii Independent increments: for $t_1 \leq t_2 \leq t_3 \leq t_4$

$$\prec (w_{t_4} - w_{t_3})(w_{t_2} - w_{t_1}) \succ = \sigma^2 \sum_n r_n [t_4 \wedge t_2 - t_4 \wedge t_1 - t_3 \wedge t_2 + t_3 \wedge t_1] = 0$$

iii Gaussian structure of the characteristic function :

$$\prec e^{\imath \sum_{k=1}^{m} \sum_{n=0}^{\infty} c_n \lambda_k [\psi_n(t_k) - \psi_n(t_{k-1})]} \succ = e^{-\frac{1}{2} \sum_n r_n \{\sum_{k=1}^{m} \lambda_k [\psi_n(t_k) - \psi_n(t_{k-1})]\}^2}$$

We use the identity

$$\sum_{k=1}^{m} \lambda_k [\psi_n(t_k) - \psi_n(t_{k-1})] =$$
$$\sum_{k=1}^{m-1} (\lambda_k - \lambda_{k+1})\psi_n(t_k) + \lambda_m \, \psi_n(t_m) - \lambda_1 \, \psi_n(t_0) := \sum_{k=0}^{m} \Delta_k \, \psi_n(t_k)$$

Thus

$$\prec e^{\imath \sum_{k=1}^{m} \sum_{n=0}^{\infty} c_n \lambda_k [\psi_n(t_k) - \psi_n(t_{k-1})]} \succ = e^{-\frac{1}{2} \sum_n r_n [\sum_{k=1}^{m} \Delta_k \, \psi_n(t_k)]^2}$$

Now we observe that

$$\sum_n r_n [\sum_{k=1}^{m} \Delta_k \, \psi_n(t_k)]^2 =$$
$$\sum_{k,l=1}^{m} \sum_n r_n \Delta_k \, \Delta_l \, \psi_n(t_k)\psi_n(t_l) = \sum_{k,l=1}^{m} \Delta_k \, \Delta_l R(t_k \, , t_l) = \prec \left(\sum_{l=0}^{m} \Delta_l \, w_{t_l} \right)^2 \succ$$

Finally using the definition of the Δ_l coefficients

$$\sum_n r_n [\sum_{k=1}^{m} \Delta_k \, \psi_n(t_k)]^2 = \prec \left[\sum_{k=1}^{m} \lambda_k (w_{t_k} - w_{t_{k-1}}) \right]^2 \succ = \sum_{k=1}^{m} \prec \lambda_k^2 (w_{t_k} - w_{t_{k-1}})^2 \succ$$

3.1 Explicit construction of the Karhunen-Loève basis

We need to solve:

$$\int_0^T \frac{ds}{T}\, R(t,s)\, \psi_n(s) = r_n\, \psi_n(t)$$

or equivalently

$$\int_0^t \frac{ds}{T} ds\, \sigma^2\, s\, \psi_n(s) + \sigma^2\, t \int_t^T \frac{ds}{T}\, \psi_n(s) = r_n\, \psi_n(t)$$

The first derivative is

$$\sigma^2 \int_t^T \frac{ds}{T}\, \psi_n(s) = r_n\, \dot{\psi}_n(t) \qquad \Rightarrow \qquad \dot{\psi}_n(T) = 0$$

and the second derivative

$$-\frac{\sigma^2}{T}\psi_n(t) = r_n\, \ddot{\psi}_n(t) \qquad \Rightarrow \qquad \ddot{\psi}_n(t) + \frac{\sigma^2}{T\, r_n}\psi_n(t) = 0$$

Thus the problem is equivalent to solving the differential equation

$$\ddot{\psi}_n(t) + \frac{\sigma^2}{T\, r_n}\psi_n(t) = 0$$

with boundary conditions

$$\psi_n(0) = \dot{\psi}_n(T) = 0$$

The boundary condition in zero yields (the factor $\sqrt{2}$ comes from unit normalization)

$$\psi_n(t) = \sqrt{2} \sin\left(\sqrt{\frac{\sigma^2}{T\, r_n}}\, t\right)$$

Imposing the condition for $t = T$ gives the "*quantization*" condition for the eigenvalues:

$$\sqrt{\frac{\sigma^2\, T}{r_n}} = (2\, n + 1)\, \frac{\pi}{2} \qquad n = 0, 1, \ldots$$

The conclusion is that the explicit Karhunen-Loève representation of the covariance of the Wiener process is

$$R(t,s) = \sum_{n=0}^{\infty} \frac{\sigma^2\, T\, \psi_n(t)\psi_n(s)}{\left(n + \frac{1}{2}\right)^2\, \pi^2}$$

4 Final remark

An example of concrete application of the Karhunen-Loève representation can be found in [4].

References

[1] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, http://math.berkeley.edu/~evans/

[2] R. Courant, and D. Hilbert, *Methods of Mathematical Physics, Vol1* Wiley, (1989).

[3] A. N. Kolmogorov, S. V. Fomin *Elements of the Theory of Functions and Functional Analysis* Dover Publications, (1999).

[4] T. Sapsis, and P. Lermusiaux, *Dynamically Orthogonal field equations for continuous stochastic dynamical systems*, Physica D, **238** 2347 (2009)and http://web.mit.edu/sapsis/www/papers/DO_fld_eqs.pdf