1 Introduction

The content of these notes is also covered by chapter 3 section B of [1].

2 Diffusion equation and central limit theorem

Consider a sequence $\{\xi_i\}_{i=1}^{\infty}$ i.i.d. $\xi_i \stackrel{d}{=} \xi$ with

$$\xi \,:\, \Omega \to \{-Dx\,,0\,,Dx\}$$

with

$$P(\xi = Dx) = T_+$$
$$P(\xi = -Dx) = T_-$$

Obviously

$$\prec \xi \succ = Dx\,(T_+ - T_-)$$
$$\prec (\xi - \prec \xi \succ)^2 \succ = (Dx)^2\,[(T_+ + T_-) - (T_+ - T_-)^2]$$

are finite as well as moments of any order. If we define

$$S_n[\xi] = \sum_{i=1}^{n} \frac{\xi_i}{n} \tag{2.1}$$

we can apply the strong law of large numbers

$$P\left(\lim_{n\uparrow\infty} S_n \to \prec \xi \succ\right) = 1$$

and the central limit theorem

$$P\left(x_- \leq \frac{S_n - \prec \xi \succ}{\sqrt{\prec (S_n - \prec \xi \succ)^2 \succ}} \leq x_+\right) \stackrel{n\uparrow\infty}{\to} \int_{x_-}^{x_+} dx\, g_{0\,1}(x)$$

with

$$\prec\, (S_n - \prec \xi \succ)^2 \succ = (Dx)^2\,[(T_+ + T_-) - (T_+ - T_-)^2]$$

The central limit theorem can be used to construct the continuum limit of the above random walk. In order to do so, we define time and the position coordinates according as $(i\,dx\,,n\,Dt) = (x\,,t)$ and we set

$$T_+ = \frac{1 + b_+\,\frac{Dx}{\sigma^2}}{2}$$
$$T_- = \frac{1 - b_-\,\frac{Dx}{\sigma^2}}{2}$$

Rather than on (2.1) we will focus on

$$W_n[\xi] := n\,S_n[n]$$

for which we get into

$$\lim_{\substack{n\uparrow\infty \\ n(Dx)^2\to\sigma^2 t}} \prec W_n[\xi] \succ = \lim_{\substack{n\uparrow\infty \\ n(Dx)^2\to\sigma^2 t}} n\,Dx^2\,\frac{b_+ + b_-}{2} = b\,t := \prec w_t \succ$$

and

$$\lim_{\substack{n\uparrow\infty \\ n(Dx)^2\to\sigma^2 t}} \prec (W_n[\xi] - n \prec \xi \succ)^2 \succ =$$
$$\lim_{\substack{n\uparrow\infty \\ n(Dx)^2\to\sigma^2 t}} n(Dx)^2 \left[1 + \frac{b_+ - b_-}{2}\frac{Dx}{\sigma^2} - \frac{(b_+ - b_-)^2\,(Dx)^2}{4\,\sigma^4}\right] = \sigma^2\,t := \prec (w_t - \prec w_t \succ)^2 \succ$$

Finally upon setting

$$\begin{aligned} \Delta_n &:= \sqrt{\prec (W_n[\xi] - n \prec \xi \succ)^2 \succ} \\ x_\pm &:= \frac{\tilde{x}_\pm}{\Delta_n} \end{aligned} \tag{2.2}$$

we have

$$\begin{aligned} &\lim_{\substack{n\uparrow\infty \\ n(Dx)^2\to\sigma^2 t}} P\left(\tilde{x}_- \leq W_n[\xi] - n \prec \xi \succ \leq \tilde{x}_+\right) \\ &= \lim_{\substack{n\uparrow\infty \\ n(Dx)^2\to\sigma^2 t}} \int_{\frac{\tilde{x}_-}{\Delta_n}}^{\frac{\tilde{x}_+}{\Delta_n}} dx\, g_{0\,,1}(x) = \int_{\tilde{x}_-}^{\tilde{x}_+} dx\, g_{0\,,\sqrt{\sigma^2 t}}(x) \end{aligned} \tag{2.3}$$

We conclude that the family of random variables

$$w_t \,:\, \Omega \times \mathbb{R}_+ \,\to\, \mathbb{R}$$

defined by the limit is distributed according to the probability density:

$$p_{w_t}(x) = \frac{e^{-\frac{(x - b\,t)^2}{2\,\sigma^2\,t}}}{\sqrt{2\,\pi\,\sigma^2\,t}}$$

The characteristic function is

$$\check{p}_{w_t}(q) = e^{\imath\,q\,b\,t - \frac{\sigma^2\,t\,q^2}{2}}$$

and satisfies the equation

$$\partial_t \check{p}_{w_t}(q) = -\frac{\sigma^2\,q^2}{2}\check{p}_{w_t}(q) + \imath\,q\,b\,\check{p}_{w_t}(q)$$

which is nothing else than the Fourier transform of the Fokker-Planck equation for constant drift and diffusion coefficients.

Remark 2.1. It worth emphasizing that the drift coefficient

$$b = \frac{b_+ + b_-}{2}$$

does not depends upon the walker standstill probability. The fact has an intuitive interpretation from the point of view of the continuum limit: we are attributing zero probability weigth to Lebesgue zero measure sets.

2.1 Independence of the increments

Let us consider the difference

$$D_m[\xi] = W_{n+m}[\xi] - W_n[\xi] = \sum_{i=n+1}^{m} \xi_i$$

clearly for any finite n and any events A, B belonging to the σ-algebra induced by W_{n+m}

$$P\left(W_n[\xi] \in A\,, D_m[\xi] \in B\right) = P\left(W_n[\xi] \in A\right) P\left(D_m[\xi] \in B\right)$$

The property must survive in the continuum limit for $n\,, m\, \in\, \mathbb{N}$ tending to infinity:

$$\begin{cases} W_{n+m}[\xi] \stackrel{(n+m)\uparrow\infty}{\rightarrow} w_{t'} \\ W_n[\xi] \stackrel{n\uparrow\infty}{\rightarrow} w_t \end{cases} \quad \Rightarrow \quad t' \geq t \quad \& \quad P(w_{t'} - w_t \in A\,, w_t \in\, B) = P(w_{t'} - w_t \in A)\, P(w_t \in\, B)$$

3 Brownian motion

Definition 3.1 (*Stochastic process*)**.** *A collection of random variables* $\{\boldsymbol{\xi}_t | t\, \geq\, 0\}$

$$\boldsymbol{\xi}\,:\, \Omega \times \mathbb{R}_+ \,\rightarrow\, \mathbb{R}^d$$

is called a stochastic process.

Realizations of stochastic process are now paths rather than numbers:

Definition 3.2 (*Sample path*)**.** *For each* $\omega\, \in\, \Omega$ *the mapping*

$$t\, \rightarrow\, \boldsymbol{\xi}_t(\omega)$$

is called the sample path of the stochastic process.

In most applications stochastic processes are characteized by means of the family of all *finite dimensional joint distributions associated* to them. This means that for a stochastic process valued on $\mathbb{R}$, for any discrete sequence $\{t_i\}_{i=1}^n$ we consider the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ and $B_1\,, \ldots\,, B_n\, \mathcal{B}$ and consider

$$P_{\xi_t}(B_1\,, t_1\,, \ldots\,, B_n\,, t_n) \equiv P(\xi_{t_1}\, \in\, B_1\,, \ldots\,, \xi_{t_n}\, \in\, B_n)$$

The so defined families of joint probability yield a consistent description of a stochastic process

$$\boldsymbol{\xi}\,:\, \Omega \times \mathbb{R}_+ \,\rightarrow\, \mathbb{R}^d$$

if the following Kolmogorov consistency conditions are satisfied

i $P(\mathbb{R}^d, t)$ for any t

ii $P_{\xi_t}(B_1\,, t_1\,, \ldots\,, B_n\,, t_n) \geq\, 0$

iii $P_{\xi_t}(B_1\,, t_1\,, \ldots\,, B_n\,, t_n) = P_{\xi_t}(B_1\,, t_1\,, \ldots\,, B_n\,, t_n\,, \mathbb{R}^d, t_{m+1})$

iv $P_{\xi_t}(B_{\pi(1)}\,, t_{\pi(1)}\,, \ldots\,, B_{\pi(n)}\,, t_{\pi(n)}) = P_{\xi_t}(B_1\,, t_1\,, \ldots\,, B_n\,, t_n\,, \mathbb{R}^d, t)$

The above definitions allow us to characterize the Wiener process as a stochastic process:

Definition 3.3 (*Brownian motion*)**.** *A real valued stochastic process*

$$w_t : \Omega \times \mathbb{R}_+ \to \mathbb{R}$$

is called a **Brownian motion** *or* **Wiener process** *if*

i $w_0 = 0$

ii *the increment* $w_t - w_s$ *has Gaussian PDF* $g_{0,\sqrt{\sigma^2 (t-s)}}(x)$ *for all* $t \geq s \geq 0$ *and* $\sigma > 0$ *a constant diffusion (volatility) coefficient.*

iii *For all times*

$$t_1 < t_2 < \ldots \leq t_n$$

the random variables

$$w_{t_1}, w_{t_2} - w_{t_1}, \ldots, w_{t_n} - w_{t_{n-1}}$$

are independent the process has independent increments).

3.1 Consequences of the definition

Some observations are in order

- It is not restrictive to consider the one dimensional case. The PDF of Brownian motion on $\mathbb{R}^d$ is obtained by multiplying PDF's

$$p_{\boldsymbol{w}_t}(\boldsymbol{x}) = \prod_{i=1}^{d} p_{w_t^i}(x_i)$$

- By i and ii we have that

$$p_{w_t}(x) = \frac{e^{-\frac{x^2}{2\sigma^2 t}}}{(2\pi\sigma^2 t)^{\frac{1}{2}}} \qquad \& \qquad p_{w_{t_2}-w_{t_1}}(x) = \frac{e^{-\frac{x^2}{2\sigma^2 (t_2 - t_1)}}}{[2\pi\sigma^2 (t_2 - t_1)]^{\frac{1}{2}}} \quad t_2 > t_1 \tag{3.1}$$

By iii The joint probability of w_{t_1} and $w_{t_2} - w_{t_1}$ is

$$p_{w_{t_1}, w_{t_2} - w_{t_1}}(x_1, y) = \frac{e^{-\frac{x_1^2}{2\sigma^2 t_1}}}{(2\pi\sigma^2 t_1)^{\frac{1}{2}}} \frac{e^{-\frac{y^2}{2\sigma^2 (t_2 - t_1)}}}{[2\pi\sigma^2 (t_2 - t_1)]^{\frac{1}{2}}}$$

By definition of probability density we can also write

$$p_{w_{t_1}, w_{t_2} - w_{t_1}}(x_1, y) = p_{w_{t_1}, w_{t_2} - x_1}(x_1, y) = p_{w_{t_1}, w_{t_2}}(x_1, y + x_1)$$

since

$$w_{t_2} = (w_{t_2} - w_{t_1}) + w_{t_1}$$

Recalling the definition of *conditional probability* we must also have

$$p_{w_{t_1}, w_{t_2}}(x_1, y + x_1) = p_{w_{t_2}|w_{t_1}}(x_1 + y, t_2 \,|\, x_1, t_1)\, p_{w_{t_1}}(x_1) \quad \forall x_1, x_2, t_2 > t_1$$

whence

$$p_{w_{t_2}|w_{t_1}}(x_1+y, t_2 \,|\, x_1, t_1) = \frac{p_{w_{t_1},w_{t_2}}(x_1, y+x_1)}{p_{w_{t_1}}(x_1)} = \frac{p_{w_{t_1},w_{t_2}-w_{t_1}}(x_1, y)}{p_{w_{t_1}}(x_1)} = \frac{e^{-\frac{y^2}{2\,\sigma^2\,(t_2-t_1)}}}{[2\,\pi\,\sigma^2\,(t_2-t_1)]^{\frac{1}{2}}}$$

Finally, upon setting $x_2 = y + x_1$ we get into:

$$p_{w_{t_2}|w_{t_1}}(x_2, t_2 \,|\, x_1, t_1) = \frac{e^{-\frac{(x_2-x_1)^2}{2\,\sigma^2\,(t_2-t_1)}}}{[2\,\pi\,\sigma^2\,(t_2-t_1)]^{\frac{1}{2}}}$$

References

[1] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, `http://math.berkeley.edu/~evans/`