1 Introduction

The lecture notes complement sections H and I of chapter 2 and appendix B of [1].

2 Four inequalities

The three inequality below are commonly used in probability and the theory of stochastic processes.

Proposition 2.1 (*Cauchy-Buniakovskii inequality*)**.** *Let ξ and η, two Borel measurable random variables with finite second moment. Then*

$$\prec |\xi\,\eta| \succ \leq \left(\prec \eta^2 \succ\right)^{1/2} \left(\prec \xi^2 \succ\right)^{1/2}$$

Proof. By definition

$$\prec |\eta\,\xi| \succ \equiv \int_{\Omega} dP(\omega)\, |\xi(\omega)\,\eta(\omega)|$$

Set

$$\tilde{\eta} = \frac{\eta}{\sqrt{\prec \eta^2 \succ}} \qquad \& \qquad \tilde{\xi} = \frac{\xi}{\sqrt{\prec \xi^2 \succ}}$$

then

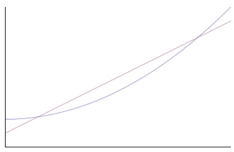
$$0 \leq \prec (|\tilde{\xi}| - |\tilde{\eta}|)^2 \succ \qquad \Rightarrow \qquad 2 \prec |\tilde{\eta}\,\tilde{\xi}| \succ \leq \prec |\tilde{\eta}|^2 \succ + \prec |\tilde{\xi}|^2 \succ = 1$$

whence the claim. □

Definition 2.1 (*Convexity*)**.** *A Borel (-measurable) function is sadid to be*

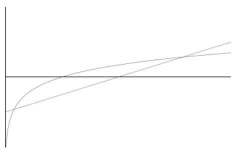
- *convex if for any two points x and y in its domain of definition Ω and any $t \in [0,1]$*

$$f(t\,x + (1-t)\,y) \leq t\,f(x) + (1-t)\,f(y)$$



- *concave if for any two points x and y in its domain of definition Ω and any $t \in [0,1]$*

$$f(t\,x + (1-t)\,y) \geq t\,f(x) + (1-t)\,f(y)$$



An example of concave function is the logarithm:

$$\ln(t\,x + (1-t)\,y) \leq t\,\ln x + (1-t)\ln y$$

Proposition 2.2 (*Jensen's inequality*)**.** *Let the Borel function $f(x)$ be downward convex and ξ a random variable with absolutely convergent first moment. Then*

$$f(\prec \xi \succ) \leq \prec f(\xi) \succ$$

Proof. Using the definition of convex function for each $x_o \in \mathbb{R}$ we can find a number $g(x_o)$ such that

$$f(x) \;\geq\; f\,(x_o) + g(x_o)\,(x - x_o)$$

The identifications $x = \xi$ and $x_o = \prec \xi \succ$ yield the proof ☐

Jensen's inequality is important for moment estimates. We have for example

$$\prec x^2 \succ^2 \leq \prec x^4 \succ$$

and more generally

Proposition 2.3 (*Lyapunov's inequality*)**.** *Let* $0 \,<\, s \,<\, t$ *then*

$$\prec |\xi|^s \succ^{1/s} \leq \prec |\xi|^t \succ^{1/t}$$

Proof. Define $r = t/s$ and

$$\eta = |\xi|^s$$

Since $r \,>\, 1$ the function $f(x) = x^r$ is convex. Jensen's inequality

$$f(\prec \eta \succ) \;\leq \prec f(\eta) \succ$$

made explicit for the present case, yields

$$\prec \eta \succ^r \leq \prec \eta^r \succ$$

whence the claim. ☐

Proposition 2.4 (*Hölder's inequality*)**.** *Let* $1 \,<\, p\,, q \,<\, \infty$ *two numbers such that*

$$\frac{1}{p} + \frac{1}{q} = 1$$

If $\prec |\xi|^p \succ$ *and* $\prec |\eta|^q \succ$ *are finite then*

$$\prec |\xi\,\eta| \succ \leq (\prec |\xi|^p \succ)^{1/p} (\prec |\eta|^q \succ)^{1/q}$$

Proof. Set

$$\tilde{\xi} = \frac{\xi}{(\prec |\xi|^p \succ)^{1/p}} \qquad \& \qquad \tilde{\eta} = \frac{\eta}{(\prec |\eta|^p \succ)^{1/p}}$$

The concavity of the logarithm

$$\ln(t\,x + (1-t)\,y) \;\geq\; t \ln x + (1-t)\;\ln y$$

implies that for any $x\,, y \,>\, 0$

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}$$

Setting

$$x = |\tilde{\xi}| \qquad \& \qquad y = |\tilde{\eta}|$$

yields

$$\prec |\tilde{\xi}\tilde{\eta}| \succ \leq 1$$

which proves the claim. ☐

3 Martingales

A nice presentation of finite dimensional martinagales is give in paragraph 11 of chapter 2 of [2].

Definition 3.1 (*Martingale*)**.** *Let $\{\xi_i\}_{i=1}^{\infty}$ a sequence of real-valued random variables, such that*

$$\prec |\xi_i| \succ < \infty \qquad \forall i$$

and $\{\mathcal{F}_k\}_{k=1}^{\infty}$ a growing sequence of σ-algebras (so that $\mathcal{F}_k \subseteq \mathcal{F}_{k'}$ if $k \leq k^{'}$), such that each of the ξ_{i+1} is $\mathcal{F}_i$-measurable. If

$$\xi_k = \prec \xi_j | \mathcal{F}_k \succ \qquad a.s \qquad \forall\, j \geq k$$

holds true we say that $\{\xi_i\}_{i=1}^{\infty}$ is a (discrete) martingale.

Examples of martingales:

- Let $\{\xi_i\}_{i=1}^{\infty}$ a sequence of i.i.d. Bernoulli random variables $\xi_i \overset{d}{=} \xi : \Omega \to \{-x, x\}$ and

  $$P_{\xi}(x) = p$$

  then

  $$S_k = \sum_{i=1}^{k} \xi_i - (2\,p - 1)\,x\,k$$

  is a martingale with respect to the sequence of σ-algebras with elements $\mathcal{F}_k$ generated by $\{\xi_i\}_{i=1}^{k}$:

  $$\begin{aligned} &\mathcal{F}_1 = \{F_x, F_{-x}\} \\ &\mathcal{F}_2 = \{F_{x,x}, F_{-x,x}, F_{x,-x}, F_{x,x}\} \\ &\mathcal{F}_3 = \{F_{x,x,x}, F_{-x,x,x}, F_{x,-x,x}, F_{x,x,-x}, F_{-x,-x,x}, F_{x,-x,-x}, F_{-x,x,-x}, F_{-x,-x,-x}\} \\ &etc. \end{aligned}$$

  Namely

  $$\prec S_{k+1} | \mathcal{F}_k \succ = \sum_{i=1}^{k-1} \xi_i - (2\,p - 1)\,x\,k + \prec \xi_i \succ = \sum_{i=1}^{k-1} \xi_i - (2\,p - 1)\,x\,(k - 1) = S_{k-1}$$

- Let $\{\xi_i\}_{i=1}^{\infty}$ a sequence of i.i.d. Bernoulli random variables $\xi_i \overset{d}{=} \xi : \Omega \to \{-1, 1\}$ and

  $$P_{\xi}(x) = p$$

  then the sequence of the

  $$\eta_i = \left(\frac{1-p}{p}\right)^{S_i}$$

  with

  $$S_i = \sum_{i=1}^{k} \xi_i$$

is a martingale with respect to the same sequence of σ-algebras as in the previous example:

$$\prec \eta_{k+1}|\mathcal{F}_k \succ = \left(\frac{1-p}{p}\right)^{S_k} \prec \left(\frac{1-p}{p}\right)^{\xi_{k+1}} \succ$$
$$= \left(\frac{1-p}{p}\right)^{S_k} \left[\left(\frac{1-p}{p}\right) p + \left(\frac{p}{1-p}\right)(1-p)\right] = \left(\frac{1-p}{p}\right)^{S_k}$$

Definition 3.2 (*Sub-Martingale and Super-Matingale*). *Let $\{\xi_i\}_{i=1}^{\infty}$ a sequence of real-valued random variables, such that*

$$\prec |\xi_i| \succ < \infty \qquad \forall i$$

and $\{\mathcal{F}_k\}_{k=1}^{\infty}$ a growing sequence of σ-algebras (so that $\mathcal{F}_k \subseteq \mathcal{F}_{k'}$ if $k \leq k'$), such that each of the ξ_{i+1} is $\mathcal{F}_i$-measurable. We call $\{\xi_i\}_{i=1}^{\infty}$ a sub-martingale if

$$\xi_k \leq \prec \xi_j|\mathcal{F}_k \succ \qquad \forall j \geq k$$

we call $\{\xi_i\}_{i=1}^{\infty}$ super-martingale if

$$\xi_k \geq \prec \xi_j|\mathcal{F}_k \succ \qquad \forall j \geq k$$

An example of sub-maringale is provided by following proposition.

Proposition 3.1. *If $\{\xi_i, \mathcal{F}_i\}_{i=1}^{\infty}$ is a martingale then $\{|\xi_i|, \mathcal{F}_i\}_{i=1}^{\infty}$ is a sub-martingale*

Proof.

$$|\xi_i| = | \prec \xi_{i+1}|\mathcal{F}_i \succ | \leq \prec |\xi_{i+1}|\,|\mathcal{F}_i \succ$$

□

Theorem 3.1 (*Discrete martingale inequalities*).

- *If $\{\xi_i\}_{i=1}^{\infty}$ is a submartingale, then*

$$P(\max_{1\leq k\leq n} \xi_k \geq x) \leq \frac{1}{x} \prec \xi_n \vee 0 \succ$$

- *If $\{\xi_i\}_{i=1}^{\infty}$ is a martingale and $1 < p < \infty$ then*

$$\prec \left(\max_{1\leq k\leq n} |\xi_k|\right)^p \succ \leq \left(\frac{p}{p-1}\right)^p \prec |\xi_k|^p \succ$$

Proof.

- The event

$$A = \left\{\omega \in \Omega | \max_{1\leq k\leq n} \xi_k(\omega) \geq x\right\} = \cup_{k=1}^{n} \{\omega \in \Omega | \xi_k(\omega) \geq x\}$$

admits the decomposition into *disjoint* events

$$A = \sum_{k=1}^{n} A_k \qquad \text{with} \qquad A_k = \cap_{l=1}^{k-1} \{\omega \in \Omega | \xi_l(\omega) < x\} \cap \{\omega \in \Omega | \xi_k(\omega) \geq x\}$$

The decomposition can be formally proven observing that

$$\{\omega\in\Omega|\xi_1(\omega)\geq x\}\cup\{\omega\in\Omega|\xi_2(\omega)\geq x\}$$
$$=\{\omega\in\Omega|\xi_1(\omega)\geq x\}\cup[\{\omega\in\Omega|\xi_2(\omega)\geq x,\xi_1(\omega)\geq x\}\cup\{\omega\in\Omega|\xi_2(\omega)\geq x,\xi_1(\omega)<x\}]$$

by *associativity* of the set union operation can be re-written as

$$\{\omega\in\Omega|\xi_1(\omega)\geq x\}\cup\{\omega\in\Omega|\xi_2(\omega)\geq x\}$$
$$=[\{\omega\in\Omega|\xi_1(\omega)\geq x\}\cup\{\omega\in\Omega|\xi_2(\omega)\geq x,\xi_1(\omega)\geq x\}]\cup\{\omega\in\Omega|\xi_2(\omega)\geq x,\xi_1(\omega)<x\}$$

but

$$\{\omega\in\Omega|\xi_2(\omega)\geq x,\xi_1(\omega)\geq x\}\subseteq\{\omega\in\Omega|\xi_1(\omega)\geq x\}\Rightarrow$$
$$\{\omega\in\Omega|\xi_1(\omega)\geq x\}\cup\{\omega\in\Omega|\xi_2(\omega)\geq x,\xi_1(\omega)\geq x\}=\{\omega\in\Omega|\xi_1(\omega)\geq x\}$$

so that

$$\{\omega\in\Omega|\xi_1(\omega)\geq x\}\cup\{\omega\in\Omega|\xi_2(\omega)\geq x\}$$
$$=\{\omega\in\Omega|\xi_1(\omega)\geq x\}+\{\omega\in\Omega|\xi_2(\omega)\geq x,\xi_1(\omega)<x\}$$

etc. For any of the A_k by Chebyshev inequality we can write

$$P(A_k)\leq\prec\frac{\xi_k}{x}\chi_{A_k}\succ$$

so that

$$\sum_{k=1}^{n}P(A_k)=P(A)\leq\sum_{k=1}^{n}\prec\frac{\xi_k}{x}\chi_{A_k}\succ$$

By definition of sub-martingale we have

$$P(A)\leq\frac{1}{x}\sum_{k=1}^{n}\prec\prec\xi_n|\mathcal{F}_k\succ\chi_{A_k}\succ=\frac{1}{x}\sum_{k=1}^{n}\prec\prec\xi_n\chi_{A_k}|\mathcal{F}_k\succ\succ$$

But

$$\prec\prec\xi_n\chi_{A_k}|\mathcal{F}_k\succ\succ=\prec\xi_n\chi_{A_k}\succ$$

so that

$$P(A)\leq\frac{1}{x}\prec\xi_n\chi_A\succ=\frac{1}{x}\prec(\xi_n\vee 0)\chi_A\succ\leq\frac{1}{x}\prec(\xi_n\vee 0)\succ$$

The first equality holds because $x>0$, the second inequality because we are adding positive terms.

- By definition

$$P(\max_{1\leq k\leq n}|\xi_k|\geq x)=1-F_{\max_{1\leq k\leq n}\xi_k}(x)=\tilde{F}_{\max_{1\leq k\leq n}\xi_k}(x)$$

so that

$$\prec\left(\max_{1\leq k\leq n}|\xi_k|\right)^p\succ=$$
$$\int_0^\infty dF_{\max_{1\leq k\leq n}|\xi_k|}(x)\,x^p=-\int_0^\infty d\tilde{F}_{\max_{1\leq k\leq n}|\xi_k|}(x)\,x^p=p\int_0^\infty dx\,P(\max_{1\leq k\leq n}|\xi_k|\geq x)\,x^{p-1}$$

But since the absolute value of a (sub)-martingale is a sub-martingale

$$P(\max_{1\le k\le n} \xi_k \ge x) \;\le\; \frac{1}{x} \prec |\xi_n|\, \chi_{\max_{1\le k\le n} \xi_k \ge x} \succ$$

we have

$$\prec \left(\max_{1\le k\le n} |\xi_k| \right)^p \succ \le p \int dx\, x^{p-2} \;\prec |\xi_n|\, \chi_{\max_{1\le k\le n} |\xi_k| \ge x} \succ$$

Regarding $|\xi_n|$ as a function $f_{|\xi_n|}(\bullet)$ of $\max_{1\le k\le n} |\xi_k|^p$ (they are measurable with respect to the same σ-algebra), we continue by writing

$$\prec \left(\max_{1\le k\le n} |\xi_k| \right)^p \succ \le\; p \int_\Omega dP\, f_{|\xi_n|}(x) \int_0^x dy\, y^{p-2} = \frac{p}{p-1} \int_\Omega dP\, f_{|\xi|_n}(x)\, x^{p-1}$$

whence finally by Hölder's inequality

$$\prec \left(\max_{1\le k\le n} |\xi_k| \right)^p \succ \le \left(\frac{p}{p-1} \right) \left[\int_\Omega dP\, f_{|\xi|_n}(x)^p \right]^{\frac{1}{p}} \left[\int_\Omega dP\, x^p \right]^{1-\frac{1}{p}}$$

□

References

[1] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, `http://math.berkeley.edu/~evans/`

[2] A. N. Shiryaev, *Probability*, 2nd Ed. Springer (1996),
`http://books.google.com/`