Exercise set 5

Exercise 1

Consider the stochastic differential equation (Ornstein-Uhlenbeck process)

$$d\xi_t = -\frac{\xi_t}{\tau} dt + \bar{x} \frac{dw_t}{\sqrt{\tau}} \tag{0.1a}$$

$$\xi_{t_o} = x_o \tag{0.1b}$$

for $\bar{x}\,, \tau\ > 0$ some given constant parameters.

- Find the fundamental solution of (0.1a) (i.e. solve (0.1a) for the arbitrary initial data $(x_o\,, t_o)$).
- Write the forward Kolmogorov equation associated to (0.1a).
- Write the transition probability density solution of the forward Kolmogorov equation satisfying boundary conditions

$$\lim_{t \downarrow t_o} p_\xi\,(x\,, t|x_o, t_o) = \delta(x - x_o) \tag{0.2a}$$

$$\lim_{|x| \uparrow \infty} |x|^{1+\varepsilon}\, p_\xi\,(x\,, t|x_o, t_o) = 0 \tag{0.2b}$$

- Analyze the structure of $p_\xi\,(x\,, t|x_o, t_o)$. In particular:
  - what is behavior under the simultaneous transformation

    $$t \to t + t' \qquad \& \qquad t_o \to t_o + t'$$

    Explain the result in terms of the properties of the drift and diffusion fields of (0.1a).
  - evaluate $\prec \xi_t \succ$
  - evaluate $\prec (\xi_t - \prec \xi_t \succ)^2 \succ$
- Study the limit

  $$\lim_{t \uparrow \infty} p_\xi\,(x\,, t|x_o, t_o)$$

  is it still solution of a partial differential equation?

Exercise 2

Consider the Ito stochastic differential equation

$$d\xi_t = -\frac{\xi_t}{\tau} + \frac{\xi_t\, dw_t}{\sqrt{\tau}} \tag{0.3a}$$

$$\xi_{t_o} = x_o \tag{0.3b}$$

- Solve the forward Kolmogorov equation for the transition probability density for ξ_t defined in $\mathbb{R}_+$.
- As in exercise 1: inquire the behavior of the transition probability density under the simultaneous transformation

  $$t \to t + t' \qquad \& \qquad t_o \to t_o + t'$$

  Discuss the consequences of your findings for the time boundary conditions for the backward Kolmogorov equation.
- Solve the backward Kolmogorov equation for the quantity

  $$f(x_o, t_o) = \prec \xi_t \succ$$

  for ξ_t solution of (0.3a), (0.3b).

Exercise 3

Consider the random differential equation

$$\dot{\xi}_t = -\frac{\xi_t}{\tau} + \frac{\xi_t\, f_t}{\sqrt{\tau}} \tag{0.4a}$$

$$\xi_{t_o} = x_o \tag{0.4b}$$

where $\tau > 0$ is a dimensional constant parameter and f_t is a Gaussian stochastic process with average

$$\prec f_t \succ = 0$$

and co-variance

$$\prec f_t\, f_{t'} \succ = \frac{e^{-\frac{|t-t'|}{\varepsilon\, \tau_o}}}{2\, \varepsilon\, \tau_o}$$

where τ_o is a dimensional parameter measured in the same units as t (if t has the interpretation of time, then τ is the characteristic correlation time of f_t). The parameter ε is instead a pure number.

- Explain why (0.4a) is mathematically well defined for $\varepsilon > 0$.
- Find the fundamental solution of (0.4a).
- Evaluate the n-th moment of ξ_t for $n \in \mathbb{N}$.

- Contrast the result for $\prec \xi_t^n \succ$ in the limit ε tending to zero with the analogous quantity evaluated for the solution of the *Ito* equation

$$d\xi_t = -\frac{\xi_t}{\tau} + \frac{\xi_t \, dw_t}{\sqrt{\tau}} \tag{0.5a}$$

$$\xi_{t_o} = x_o \tag{0.5b}$$

and the Stratonovich equation

$$d\xi_t = -\frac{\xi_t}{\tau} + \frac{\xi_t \, dw_t}{\sqrt{\tau}} \tag{0.6a}$$

$$\xi_{t_o} = x_o \tag{0.6b}$$

Discuss your findings: what is the meaning of taking the limit ε tending to zero for the process f_t.

Exercise 4

Given a smooth drift

$$\boldsymbol{b} : \mathbb{R}^d \to \mathbb{R}^d$$

and diffusion covariance

$$\boldsymbol{g} : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$$

We call

$$-\partial_{x^i}[b^i(\boldsymbol{x}) \; p_\star(\boldsymbol{x})] + \frac{1}{2}\partial_{x^i}\partial_{x^j}[g^{ij}(\boldsymbol{x}) \, p_\star(\boldsymbol{x})] = 0$$

the stationary forward Kolmogorov equation (Fokker-Planck) on $\mathbb{R}^d$. Suppose (0.7) admits a *normalizable* positive definite solution. Show that if there exists well defined transition probability density p_ξ on $\mathbb{R}^d$ solution of

$$\partial_t p_\xi(\boldsymbol{x}, t \,|\, \boldsymbol{x}_o, t_o) + \partial_{x^i}[b^i(\boldsymbol{x}) \; p_\xi(\boldsymbol{x}, t \,|\, \boldsymbol{x}_o, t_o)] = \frac{1}{2}\partial_{x^i}\partial_{x^j}[g^{ij}(\boldsymbol{x}) \, p_\xi(\boldsymbol{x}, t \,|\, \boldsymbol{x}_o, t_o)]$$

then

$$p_\star(\boldsymbol{x}) = \int_{\mathbb{R}^d} d^d x_o \, p_\xi(\boldsymbol{x}, t \,|\, \boldsymbol{x}_o, t_o) \, p_\star(\boldsymbol{x}_o)$$

Which are the consequences of such identity for the limit

$$\lim_{t - t_o \uparrow \infty} p_\xi(\boldsymbol{x}, t \,|\, \boldsymbol{x}_o, t_o)$$