Exercise set 3

Exercise 1

Prove that the characteristic function $\phi(t)$ of any probability distribution is positive definite, i.e. for any n complex numbers $\xi_1, \ldots, \xi_n$ and real numbers $t_1, \ldots, t_n$

$$\sum_{i,j=1}^{n} \phi(t_i - t_j)\xi_i \xi_j^* \geq 0$$

Here ξ^* denotes the complex conjugate of ξ.

Remark 0.1 (*Remark*)**.** It is also easy to see that $\phi(t)$ (using the properties of Lebesgue integral) is a uniformly continuous function of t.

Exercise 2

Let $\{\xi_i\}_{i=1}^{\infty}$ a sequence of i.i.d. *Gaussian* random variables with zero average and unit variance. Verify that for any $\lambda \in \mathbb{R}$

$$\eta_n = e^{\lambda \sum_{i=1}^{n} \xi_i - \frac{\lambda^2 n}{2}}$$

is a martingale with respect to the σ-algebra induced by the first $\{\xi_i\}_{i=1}^{m}$ variables $n \geq m$.

Exercise 3

Let $\{\xi_i\}_{i=1}^{\infty}$ a sequence of i.i.d. *uniformely distributed* random variable. Given $\lambda \in \mathbb{R}$ determine the condition that some map $c(\lambda)$ should satisfy in order

$$\eta_n = e^{\lambda \sum_{i=1}^{n} \xi_i - c(\lambda)}$$

- to be a super-martingale
- to be a martingale
- to be a sub-martingale

with respect to the σ-algebra induced by the first $\{\xi_i\}_{i=1}^{m}$ variables $n \geq m$.

Exercise 4

Let $\{w_t\,, t \geq 0\}$ a one dimensional Brownian motion with

$$\prec w_t \succ = 0 \qquad \& \qquad \prec w_t^2 \succ = \sigma^2\, t$$

Say which one of the following stochastic processes are martingales with respect to the the σ-algebra generated $\mathcal{F}_t$ generated by $\{w_s|\,, s \leq\, t\}$:

- $\{w_t\,, t \geq 0\}$.
- $\left\{w_t^2 - x\, t\,, t \geq 0\right\}$ for some suitable $x \,\in\, \mathbb{R}$.
- $\left\{e^{xt} \cos(w_t)\,, t \geq 0\right\}$ for some suitable $x \,\in\, \mathbb{R}$.

Exercise 5

Let $p_o : \mathbb{R} \to \mathbb{R}_+$ an integrable function. Consider for any $t \,\geq\, s$

$$p(x,t;s) = \int_{\mathbb{R}} dy\, \frac{e^{-\frac{(x-y)^2}{2\,\sigma^2\,(t-s)}}}{\sqrt{2\,\pi\,\sigma^2\,(t-s)}} p_o(y) \tag{0.1}$$

- Under which condition $p(x,t;s)$ is a probability density?
- Show that $p(x,t;s)$ satisfies a suitable Fokker-Planck equation (hint: use the Fourier transform).
- Determine the limit

$$\lim_{t \downarrow s} p(x,t;s)$$

- Determine an initial value problem (evolution law and inirtial condition) of which () is solution.
- Give a probabilistic interpretation of the integral kernel

$$k(x,t,y,s) = \frac{e^{-\frac{(x-y)^2}{2\,\sigma^2\,(t-s)}}}{\sqrt{2\,\pi\,\sigma^2\,(t-s)}}$$

- Give a probabilistic interpretation of the integral defining $p(x,t)$.
- Compute $p(x,t)$ explicitly in the case

$$p_o(y) = \frac{e^{-\frac{(x-y)^2}{2\,\rho^2}}}{\sqrt{2\,\pi\,\rho^2}}$$

  and determine a value of ρ^2 such that

$$\partial_s p(x,t;s) = 0$$

  and interpret the result.