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More detailed explanation of the bean machine

Wikipedia (hhttp://en.wikipedia.org/wiki/Bean_machine) outlines the reasoning beneath the bean machine. The explanation below is a bit more detailed.

The machine has n rows of bins. A ball in the kth bin of the machine ($k = 0, \ldots, n$) has bounced k times to the right with probability p and $n - k$ times to the left with probability $1 - p$. The probability of each path to the kth bin is $p^k(1 - p)^{n-k}$.

If $p = 1 - p = 0,5$ then the probability of a path to any bin is the same or $(0,5)^n$. Nevertheless, more beans end up in the bins in the middle than in the leftmost bin, say, because more paths lead to the bins in the middle.

How many paths lead to the kth bin? The number of paths can be calulated by the argument below.

Let us assume we have n objects of two kind — say of colour red and lemon. How many arrangements of the objects can be made? We will denote this figure by N. Let us assume first that we could differentiate the red objects. Then $N \times k!$ arrangements could be made because there are $k!$ different arrangements of the now separable red objects for each of the N original arrangements. Next let us assume that also the lemon objects could be differentiated. Then $N \times k! \times (n-k)!$ arrangements could be created because there are $(n - k)!$ ways to arrange the now separable lemon objects for each of the previous arrangements. But now we have assumed that all of the objects could be differentiated. Hence there must be $n!$ different arrangements altogether, and it must be the case that

$$N \times k! \times (n - k)! = n!$$

The number of arrangements N can now be solved to be

$$N = \frac{n!}{k! \times (n - k)!} = \binom{n}{k}.$$

In the argument above replace red and lemon objects by crossings to the right and left. The n crossings compose jointly a path to the kth bin with k turns to the right and $n - k$ to the left in an order determined by the arrangement. According to the argument above, the number of such arrangements or paths to the kth bin is $\binom{n}{k}$.

The paths are exclusionary and, as explained above, each path is as probable. The probability of a bean ending in the kth bin can thus be found by summing the probabilities of all the paths leading to the kth bin or by multiplying the probability of a single path by the number of paths ($k = 0, \ldots, n$):

$$\begin{aligned} \mathsf{P}(\text{bean ends up in the } k\text{th bin}) &= p^k(1-p)^{n-k} + \ldots + p^k(1-p)^{n-k} \\ &= \binom{n}{k} p^k (1-p)^{n-k}. \end{aligned}$$