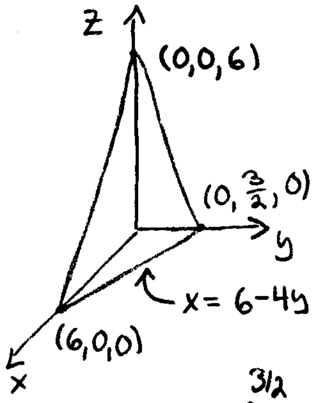
① $V = \{(x,y,z) \mid 0 \le x \le 2,\ 0 \le y \le 3$ ja $1 \le z \le 2\}$.

$$\iiint_V \frac{xy}{z}\,dxdydz = \int_0^2 dx \int_0^3 dy \int_1^2 dz\, \frac{xy}{z} = \int_0^2 dx \int_0^3 dy \Big/_{1}^{2} xy \ln z$$

$$= \int_0^2 dx \int_0^3 dy\, xy \ln 2 = \int_0^2 dx \Big/_{0}^{3} \frac{xy^2}{2} \ln 2 = \int_0^2 dx\, x\, \frac{9\ln 2}{2}$$

$$= \Big/_{0}^{2} x^2\, \frac{9 \ln 2}{4} = 9 \ln 2.$$

②



$x + 4y + z = 6 \iff z = 6 - x - 4y$.

$V = \{(x,y,z) \mid 0 \le y \le \frac{3}{2},\ 0 \le x \le 6-4y$ ja $0 \le z \le 6-x-4y\}$

$$\iiint_V dxdydz = \int_0^{3/2} dy \int_0^{6-4y} dx \int_0^{6-x-4y} dz$$

$$= \int_0^{3/2} dy \int_0^{6-4y} dx \Big/_{0}^{6-x-4y} z = \int_0^{3/2} dy \int_0^{6-4y} dx\, (6-x-4y)$$

$$= \int_0^{3/2} dy \Big/_{0}^{6-4y} \left(6x - \frac{x^2}{2} - 4xy\right) = \int_0^{3/2} dy \left(36 - 24y - \frac{(6-4y)^2}{2} - 24y + 16y^2\right)$$

$$= \Big/_{0}^{3/2} \left(36y - 12y^2 + \frac{(6-4y)^3}{24} - 12y^2 + \frac{16}{3}y^3\right) = 54 - 27 + 0 - 27 + 18 - 9 = 9.$$

Integroimisjärjestyksen voi nyt valita kuudella tavalla ($6 = 3 \cdot 2 \cdot 1$). Tässä on niistä vielä kaksi.

$$\iiint_V dxdydz = \int_0^6 dx \int_0^{(6-x)/4} dy \int_0^{6-x-4y} dz = \int_0^6 dx \int_0^{(6-x)/4} dy\, (6-x-4y)$$

$$= \int_0^6 dx \Big/_{0}^{(6-x)/4} (6y - xy - 2y^2) = \frac{1}{4}\int_0^6 dx \left(36 - 6x - 6x + x^2 - \frac{(6-x)^2}{2}\right)$$

$$= \frac{1}{4} \Big/_0^6 \left(36x - 6x^2 + \frac{x^3}{3} + \frac{(6-x)^3}{6}\right) = 9,$$

$$\iiint_V dx\,dy\,dz = \int_0^6 dz \int_0^{6-z} dx \int_0^{(6-x-z)/4} dy = \ldots = 9,$$

(3) $V = \{(x,y,z) \mid 1 \le y \le 2,\ -y \le x \le y \text{ ja } x+y \le z \le x+2y\},$

$$\int_V f = \iiint_V f(x,y,z)\,dx\,dy\,dz = \int_1^2 dy \int_{-y}^{y} dx \int_{x+y}^{x+2y} dz\,(y+1)z$$

$$= \int_1^2 dy \int_{-y}^{y} dx \Big/_{x+y}^{x+2y} (y+1)\,z^2/2 = \int_1^2 dy \int_{-y}^{y} dx\,(y+1)(x^2+4xy+4y^2$$

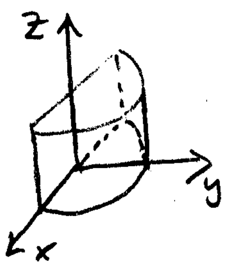
$$-x^2-2xy-y^2)/2 = \int_1^2 dy \int_{-y}^{y} dx\,(y+1)(2xy+3y^2)/2$$

$$= \int_1^2 dy \Big/_{-y}^{y} (y+1)(x^2y+3xy^2)/2 = \int_1^2 dy\,(y+1)(y^3+3y^3-y^3+3y^3)/2$$

$$= \int_1^2 dy\,(y+1)3y^3 = 3\int_1^2 dy\,(y^4+y^3) = 3\Big/_1^2 \left(\frac{y^5}{5} + \frac{y^4}{4}\right)$$

$$= 3\left(\frac{32}{5} + \frac{16}{4} - \frac{1}{5} - \frac{1}{4}\right) = 3\left(\frac{31}{5} + \frac{15}{4}\right) = 3\cdot\frac{124+75}{20} = \frac{597}{20} = 29\,\frac{17}{20}.$$

(4)



$V = \{(x,y,z) \mid x^2+y^2 \le 2^2,\ 0 \le z \le 2 \text{ ja } y \ge 0\},$

Siirrytään sylinterikoordinaatteihin (vertaa moniste).

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z \end{cases}$$

Muunnoksen Jacobin determinantti $J_g(\rho,\varphi,z) = \rho.$

$0 \le \rho = \sqrt{x^2+y^2} \le 2$ ja $0 \le \varphi \le \pi$, sillä $y \ge 0$. Nyt saadaan

$$\iiint_V ((x^2+y^2)^2+z)\,dx\,dy\,dz = \int_0^2 d\rho \int_0^2 dz \int_0^\pi d\varphi\,((\rho^2)^2+z)\underbrace{|J_g(\rho,\varphi,z)|}_{=\rho}$$

$$= \pi \int_0^2 d\rho \int_0^2 dz\,(\rho^5+\rho z) = \pi \int_0^2 d\rho \Big/_0^2 (\rho^5 z + \rho z^2/2)$$

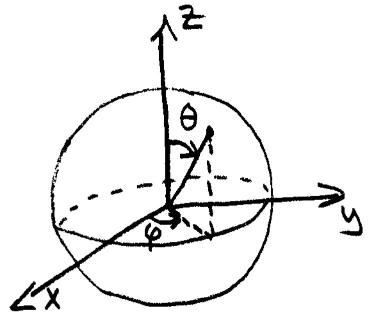
$$= 2\pi \int_0^2 d\rho\,(\rho^5+\rho) = 2\pi \Big/_0^2 \left(\frac{\rho^6}{6}+\frac{\rho^2}{2}\right) = 2\pi\left(\frac{64}{6}+2\right)$$

$$= 2\pi\left(\frac{32+6}{3}\right) = \frac{76\pi}{3},$$

5) $V = \{(x,y,z) \mid 1^2 \le x^2+y^2+z^2 \le 2^2 \text{ ja } z \ge 0\}$. Siirrytään pallokoordinaatteihin (katso verkkomoniste).

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

Muunnoksen Jacobin determinantti $J_g(r,\theta,\varphi) = r^2\sin\theta$.



$$\int_V f = \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V \frac{x^2+y^2}{x^2+y^2+z^2}\,dx\,dy\,dz$$

$$= \iiint_V \frac{x^2+y^2+z^2-z^2}{x^2+y^2+z^2}\,dx\,dy\,dz = \int_1^2 dr \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi\, \frac{r^2-r^2\cos^2\theta}{r^2}\underbrace{|J_g(r,\theta,\varphi)|}_{=r^2\sin\theta}$$

$$= 2\pi \int_1^2 dr \int_0^{\pi/2} d\theta\, r^2(\sin\theta - \cos^2\theta\sin\theta)$$

$$= 2\pi \int_1^2 dr \Big/_0^{\pi/2} r^2\left(-\cos\theta + \tfrac{1}{3}\cos^3\theta\right) = 2\pi \int_1^2 dr\, r^2\left(0-\left(-1+\tfrac{1}{3}\right)\right)$$

$$= \frac{4\pi}{3}\int_1^2 dr\, r^2 = \frac{4\pi}{3}\Big/_1^2 \frac{r^3}{3} = \frac{4\pi}{3}\left(\frac{8}{3}-\frac{1}{3}\right) = \frac{28\pi}{9}$$

⑥ $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = C/(1+x^2+y^2)^3$ on tiheysfunktio, jos ja vain jos

(i) $f(x,y) \geq 0$ kaikilla $(x,y) \in \mathbb{R}^2$ ja

(ii) $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$.

Ehto (i) toteutuu, jos ja vain jos $C \geq 0$. Siis olkoon $C \geq 0$. Tällöin voidaan soveltaa verkkomonisteessa esitettyä epäoleellisten tasointegraalien teoriaa, sillä nyt $f(x,y) \geq 0$ kaikilla $(x,y) \in \mathbb{R}^2$.

Olkoon $M > 0$. Merkitään $A_M = \{(x,y) \mid x^2+y^2 \leq M^2\}$ ja siirrytään napakoordinaatteihin (verkkomoniste)

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad J_g(r,\varphi) = r,$$

jolloin saadaan

$$\iint_{A_M} \frac{C}{(1+x^2+y^2)^3}\,dx\,dy = \iint_{[0,M]\times[0,2\pi]} \frac{C}{(1+r^2)^3}\,|J_g(r,\varphi)|\,dr\,d\varphi$$

$$= \int_0^M dr \int_0^{2\pi} d\varphi\, \frac{Cr}{(1+r^2)^3} = 2\pi \int_0^M dr\, \frac{Cr}{(1+r^2)^3}$$

$$= -\frac{\pi C}{2} \int_0^M dr\,(2r)(-2)(1+r^2)^{-3} = -\frac{\pi C}{2} \Big/_0^M (1+r^2)^{-2}$$

$$= -\frac{\pi C}{2}\left((1+M^2)^{-2} - 1\right) \xrightarrow[M\to\infty]{} \frac{\pi C}{2}.$$

Siis $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$, jos ja vain jos $C = 2/\pi$. Niinpä f on tiheysfunktio, jos ja vain jos $C = 2/\pi$.