Introduction to Fourier Analysis
Home assignment 9

1. Let

$$u(\varphi) = \int_{-1}^{1} \varphi(x,x)\,\mathrm{d}x, \qquad \varphi \in \mathscr{S}\big(\mathbb{R}^2\big)\,.$$

Show that $u \in \mathscr{S}'(\mathbb{R}^2)$ and compute its first order derivatives. What is $(\partial_1 + \partial_2)\,u$?

Solution. The inequality

$$|u(\varphi)| \leqslant \int_{-1}^{1} |\varphi(x,x)|\;\mathrm{d}x \leqslant 2\,\|\varphi\|_{\mathscr{L}^\infty(\mathbb{R}^2)}\,,$$

which holds for all $\varphi \in \mathscr{S}(\mathbb{R}^2)$, shows that $u \in \mathscr{S}'(\mathbb{R}^2)$. The first order derivatives of u are of course given by the formula

$$\partial_i u(\varphi) = -u(\partial_i\varphi) = -\int_{-1}^{1} \partial_i\varphi(x,x)\,\mathrm{d}x,$$

which holds for all $\varphi \in \mathscr{S}(\mathbb{R}^2)$ and for each $i \in \big\{1,2\big\}$.

Finally, by the chain rule, we have for each $\varphi \in \mathscr{S}(\mathbb{R}^2)$ that

$$\begin{aligned}(\partial_1+\partial_2)u(\varphi) &= -\int_{-1}^{1} \big(\partial_1\varphi(x,x) + \partial_2\varphi(x,x)\big)\,\mathrm{d}x \\ &= -\int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x}\varphi(x,x)\,\mathrm{d}x = \varphi(-1,-1) - \varphi(1,1)\,.\end{aligned}$$

That is, we have

$$(\partial_1+\partial_2)\,u = \delta_{\langle -1,-1\rangle} - \delta_{\langle 1,1\rangle}.$$

2. Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a},$$

when a is real and not equal to an integer.

Solution. Let f be the function given by the formula

$$f(x) = \begin{cases} 1-|x| & \text{if } |x| \leqslant 1, \text{ and} \\ 0, & \text{when } |x| > 1, \end{cases}$$

for all $x \in \mathbb{R}_+$. The Fourier transform of f was calculated in the exercise 5 of the seventh problem set. In the same way, it is not difficult to see that with the conventions used in the book of Stein and Shakarchi, we have

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, \mathrm{d}x = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2,$$

for nonzero real numbers ξ. Since f is even, we have $\widehat{\widehat{f}} = f$. Applying the Poisson summation formula to the function $\widehat{f}$ results in

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2 \pi a}{\pi^2\,(n+a)^2} = \sum_{n=-\infty}^{\infty} \widehat{f}(n+a) = \sum_{n=-\infty}^{\infty} f(n)\, e^{2\pi i n a} = f(1) = 1,$$

thereby recovering the required identity.

3. This exercise collects the basic properties of the gamma function.

a) The *gamma function* $\Gamma(s)$ is defined by

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1}\, \mathrm{d}x, \qquad s > 0.$$

Prove that this is well defined, i.e. that the integral exists as an improper integral.

b) Prove that $\Gamma(s+1) = s\Gamma(s)$, $s > 0$. Conclude that for every positive integer n we have $\Gamma(n+1) = n!$.

c) Show that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Solution. a) Clearly the integrand is a continuous function. In the neighbourhood of zero, the integrability is easily seen because the exponential factor is bounded and the x^{s-1}-factor is integrable. For large x the integrability is even more obvious. The integrand is positive and continuous, so the integral exists as a simple Lebesgue integral instead of a mere improper integral.

b) The dominated convergence theorem allows us to employ an integration by parts to get

$$\Gamma(s+1) = \int_0^{\infty} e^{-x} x^s\, \mathrm{d}x = \lim_{T\longrightarrow\infty} \int_{T^{-1}}^{T} e^{-x} x^s\, \mathrm{d}x$$

$$= \lim_{T\longrightarrow\infty} \left(-e^{-x} x^s \Big]_{T^{-1}}^{x=T} + s \int_{T^{-1}}^{T} e^{-x} x^{s-1}\, \mathrm{d}x \right) = s \int_0^{\infty} e^{-x} x^{s-1}\, \mathrm{d}x = s\Gamma(s)$$

for all $s \in \mathbb{R}_+$.

The claimed formula $\Gamma(n+1) = n!$ is seen to hold for all nonnegative integers n by induction once it is observed that

$$\Gamma(1) = \int_0^\infty e^{-x}\, \mathrm{d}x = \lim_{T \longrightarrow \infty} \left(-e^{-x}\right)\Big]_{T^{-1}}^{x=T} = \lim_{T \longrightarrow \infty} \left(e^{-T^{-1}} - e^{-T}\right) = e^{-0} = 1 = 0!.$$

c) A simple change of variables gives

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-x}\, \mathrm{d}x}{\sqrt{x}} = \int_0^\infty \frac{e^{-y^2} \cdot 2y\mathrm{d}y}{y} = 2\int_0^\infty e^{-y^2}\, \mathrm{d}y = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi},$$

and by part b), we have

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

4. Define the *zeta function* by

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \qquad s > 1.$$

Prove that

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2}\int_0^\infty t^{\frac{s}{2}-1}\left(\vartheta(t) - 1\right)\mathrm{d}t,$$

where the *theta function* is defined by

$$\vartheta(s) = \sum_{n=-\infty}^\infty e^{-\pi n^2 s}.$$

Solution.

$$\frac{1}{2}\int_0^\infty t^{\frac{s}{2}-1}\left(\vartheta(t) - 1\right)\mathrm{d}t = \int_0^\infty t^{\frac{s}{2}-1}\sum_{n=1}^\infty e^{-\pi n^2 t}\,\mathrm{d}t = \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t} t^{\frac{s}{2}-1}\,\mathrm{d}t$$

$$= \sum_{n=1}^\infty \int_0^\infty e^{-u}\left(\frac{u}{\pi n^2}\right)^{\frac{s}{2}-1}\frac{\mathrm{d}u}{\pi n^2} = \sum_{n=1}^\infty \pi^{-\frac{s}{2}} n^{-s}\int_0^\infty e^{-u}u^{\frac{s}{2}-1}\,\mathrm{d}u = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

5. Let's return to the X-ray transform. For each $\langle t, \vartheta\rangle \in \mathbb{R} \times [-\pi, \pi]$ let $L_{t,\vartheta}$ be the line in the $\langle x, y\rangle$-plane defined by

$$x\cos\vartheta + y\sin\vartheta = t.$$

Define the X-ray transform for $f \in \mathscr{S}(\mathbb{R}^2)$ by

$$\big(Xf\big)(t,\vartheta) = \int\limits_{L_{t,\vartheta}} f = \int\limits_{-\infty}^{\infty} f\Big(t\cos\vartheta + u\sin\vartheta\,;\; t\sin\vartheta - u\cos\vartheta\Big)\,\mathrm{d}u.$$

Compute Xg, when

$$g(x,y) = e^{-\pi(x^2+y^2)}.$$

Solution. The computation reduces to Gaussian integrals. For any $t \in \mathbb{R}$ and any $\vartheta \in [-\pi,\pi]$.

$$\begin{aligned}
\big(Xg\big)\,(t,\vartheta) &= \int\limits_{-\infty}^{\infty} g\Big(t\cos\vartheta + u\sin\vartheta\,;\; t\sin\vartheta - u\cos\vartheta\Big)\,\mathrm{d}u \\
&= \int\limits_{-\infty}^{\infty} \exp\Big(-\,\pi\,(t\cos\vartheta + u\sin\vartheta)^2 - \pi\,(t\sin\vartheta - u\cos\vartheta)^2\Big)\,\mathrm{d}u \\
&= \int\limits_{-\infty}^{\infty} \exp\Big(-\,\pi t^2\cos^2\vartheta - \pi u^2\sin^2\vartheta - 2\pi tu\cos\vartheta\sin\vartheta \\
&\qquad\qquad - \pi t^2\sin^2\vartheta - \pi u^2\cos^2\vartheta + 2\pi tu\sin\vartheta\cos\vartheta\Big)\,\mathrm{d}u \\
&= \int\limits_{-\infty}^{\infty} \exp\big(-\pi t^2 - \pi u^2\big)\,\mathrm{d}u = e^{-\pi t^2}\int\limits_{-\infty}^{\infty} e^{-\pi u^2}\mathrm{d}u = e^{-\pi t^2}.
\end{aligned}$$

6. Let again X denote the X-ray transform. Show that $f \in \mathscr{S}(\mathbb{R}^2)$ and $Xf \equiv 0$ implies $f \equiv 0$.

Solution. This follows from the important *Fourier slice theorem* which says that

$$\big(\mathscr{F}Xf\big)\,(t,\vartheta) = \sqrt{2\pi}\,\widehat{f}(t\cos\vartheta, t\sin\vartheta)$$

for all $t \in \mathbb{R}$ and $\vartheta \in [-\pi,\pi]$. Here $\mathscr{F}$ stands for taking the one-dimensional Fourier transform in the first variable. For if $Xf \equiv 0$, then $\mathscr{F}Xf \equiv 0$, so that $\widehat{f} \equiv 0$, implying that $f \equiv 0$.

The proof of the Fourier slice theorem rests on a simple linear change of integration variables. For all $t \in \mathbb{R}$ and $\vartheta \in [-\pi,\pi]$, we have

$$\begin{aligned}
\big(\mathscr{F}Xf\big)(t,\vartheta) &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f\Big(v\cos\vartheta + u\sin\vartheta\,;\; v\sin\vartheta - u\cos\vartheta\Big)\,\mathrm{d}u\,e^{-ivt}\,\mathrm{d}v \\
&= \frac{1}{\sqrt{2\pi}}\iint\limits_{\mathbb{R}^2} f(x,y)\,e^{-i(x\cos\vartheta + y\sin\vartheta)t}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{2\pi}\,\widehat{f}(t\cos\vartheta, t\sin\vartheta)\,.
\end{aligned}$$