Introduction to Fourier Analysis
Home assignment 8

1. Let $A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation, and $f \in \mathscr{L}^1(\mathbb{R}^n)$. Define $g(x) = f(Ax)$, $x \in \mathbb{R}^n$. Prove that if A is invertible, then $g \in \mathscr{L}^1(\mathbb{R}^n)$. Also, compute $\widehat{g}$.

Solution. The first claim is rather obvious, for

$$\big\|g\big\|_{\mathscr{L}^1} = \int\limits_{\mathbb{R}^n} \big|f(Ax)\big|\,\mathrm{d}x = \frac{1}{|\det A|}\int\limits_{\mathbb{R}^n} \big|f(y)\big|\,\mathrm{d}y < \infty.$$

If the matrix A was not invertible, the image of the corresponding linear operator would be a set of measure zero in $\mathbb{R}^n$, and thus g could not be well-defined.

The computation of $\widehat{g}$ is straightforward: for almost all $\xi \in \mathbb{R}^n$,

$$\begin{aligned}\widehat{g}(\xi) &= \frac{1}{\big(\sqrt{2\pi}\big)^n}\int\limits_{\mathbb{R}^n} f(Ax)\,e^{-ix\cdot\xi}\mathrm{d}x = \frac{1}{\big(\sqrt{2\pi}\big)^n\,|\det A|}\int\limits_{\mathbb{R}^n} f(y)\,e^{-iA^{-1}y\cdot\xi}\mathrm{d}y \\ &= \frac{1}{\big(\sqrt{2\pi}\big)^n\,|\det A|}\int\limits_{\mathbb{R}^n} f(y)\,e^{-iy\cdot\left(A^{-1}\right)^T\xi}\mathrm{d}y = \frac{1}{|\det A|}\widehat{f}\big((A^{-1})^T\xi\big).\end{aligned}$$

2. Give an example of a function $f \in \mathscr{L}^2$ such that $f \notin \mathscr{L}^1$ but $\widehat{f} \in \mathscr{L}^1$.

Solution. Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be the function for which for each $x \in \mathbb{R}$

$$f(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\cdot\frac{\sin x}{x}, & \text{when } x \neq 0, \text{ and} \\ \sqrt{\frac{2}{\pi}}, & \text{when } x = 0. \end{cases}$$

Furthermore, let $g = \chi_{[-1,1]}\colon \mathbb{R} \longrightarrow \mathbb{C}$.

We have $f \in \mathscr{L}^2(\mathbb{R})$ since

$$\int\limits_{-\infty}^{\infty} \left|\frac{\sin x}{x}\right|^2 \mathrm{d}x \ll 1 + \int\limits_{1}^{\infty}\frac{\mathrm{d}x}{x^2} < \infty.$$

On the other hand, in the exercise 6 of the second exercise set we proved that for any $N \in \mathbb{Z}_+$, we have

$$\int\limits_{0}^{N\pi}\frac{|\sin y|\,\mathrm{d}y}{y} \gg \log N.$$

Thus, $f \notin \mathscr{L}^1(\mathbb{R})$. For the function g, it is obvious that it belongs to both $\mathscr{L}^1(\mathbb{R})$ and $\mathscr{L}^2(\mathbb{R})$.

In the previous exercise set, in exercise 4, it was proved that $\widehat{g} = f$. Since g is an even function, we have $\widehat{f} = g$, and therefore conclude that f is an example of the required kind.

3. Let for $0 < r < \infty$,

$$g_r(x) = e^{-r|x|^2}, \qquad x \in \mathbb{R}^n.$$

Compute the convolution $g_{r_1} * g_{r_2}$.

Solution. Since $\widehat{g_{\frac{1}{2}}} = g_{\frac{1}{2}}$, we have

$$\widehat{g_r}(\xi) = \widehat{g_{\frac{1}{2}}\big(\sqrt{2r}\cdot\big)}(\xi) = \frac{1}{\big(\sqrt{2r}\big)^n}\widehat{g_{\frac{1}{2}}}\left(\frac{\xi}{\sqrt{2r}}\right) = \frac{e^{-\frac{1}{4r}|\xi|^2}}{\big(\sqrt{2r}\big)^n},$$

for all $r \in \mathbb{R}_+$ and $\xi \in \mathbb{R}^n$. Now we can easily compute the Fourier transform of the convolution $g_{r_1} * g_{r_2}$ since

$$\begin{aligned}\widehat{g_{r_1} * g_{r_2}}(\xi) = \big(\sqrt{2\pi}\big)^n\, \widehat{g_{r_1}}(\xi)\, \widehat{g_{r_2}}(\xi) &= \frac{\big(\sqrt{2\pi}\big)^n}{2^n\big(\sqrt{r_1 r_2}\big)^n} e^{-\left(\frac{1}{4r_1}+\frac{1}{4r_2}\right)|\xi|^2} \\ &= \left(\sqrt{\frac{\pi}{2r_1 r_2}}\right)^n g_{\frac{1}{4r_1}+\frac{1}{4r_2}}(\xi)\,,\end{aligned}$$

for all $\xi \in \mathbb{R}^n$. Hence

$$\begin{aligned}\big(g_{r_1} * g_{r_2}\big)(x) &= \left(\sqrt{\frac{\pi}{2r_1 r_2}}\right)^n \widehat{g_{\frac{1}{4r_1}+\frac{1}{4r_2}}}(-x) \\ &= \left(\sqrt{\frac{\pi}{2r_1 r_2}}\right)^n \frac{g_{\frac{r_1 r_2}{r_1+r_2}}(-x)}{\left(\sqrt{\frac{1}{2r_1}+\frac{1}{2r_2}}\right)^n} = \left(\sqrt{\frac{\pi}{r_1+r_2}}\right)^n e^{-\frac{r_1 r_2}{r_1+r_2}|x|^2}.\end{aligned}$$

4. Let A be a positive definite symmetric $n \times n$ matrix with real coefficients. Compute the Fourier transform of $f(x) = e^{-Ax\cdot x}$, $x \in \mathbb{R}^n$.

Solution. Let B be a positive definite symmetric $n \times n$ matrix with real coefficients which is a square root of A in the sense that $B^2 = A$. One way to see that such a matrix exists is to use the fact that the matrix A is orthogonally diagonalizable, that is, $A = O^T \Lambda O$ for some real orthogonal $n \times n$ matrix O and some real diagonal matrix Λ. Since A is positive definite, the diagonal elements of Λ are positive real numbers and there clearly exists another diagonal matrix Ξ with positive real elements such that $\Xi^2 = \Lambda$. Now we can simply set $B = O^T \Xi O$.

Using the notation introduced in the previous exercise, we have

$$\widehat{g_1} = \left(\frac{1}{\sqrt{2}}\right)^n g_{\frac{1}{4}}.$$

The idea of introducing the matrix B is that now

$$f(x) = e^{-Ax\cdot x} = e^{-Bx\cdot Bx} = g_1(Bx)$$

for all $x \in \mathbb{R}^n$. By exercise 1, the function f is in $\mathscr{L}^1$, and we have for any $\xi \in \mathbb{R}^n$ that

$$\widehat{f}(\xi) = \widehat{g_1(B\cdot)}(\xi) = \frac{g_{\frac{1}{4}}(B^{-1}\xi)}{\left(\sqrt{2}\right)^n \cdot \det B} = \frac{e^{-\frac{1}{4}B^{-1}x\cdot B^{-1}x}}{\sqrt{2^n}\cdot\sqrt{\det A}} = \frac{e^{-\frac{1}{4}A^{-1}x\cdot x}}{\sqrt{2^n \det A}}.$$

5. Let $f \in \mathscr{L}^1(\mathbb{R})$, and assume that $\widehat{f}$ is continuous and satisfies

$$\widehat{f}(\xi) = O\Big(|\xi|^{-1-\alpha}\Big)$$

as $|\xi| \longrightarrow \infty$ for some $0 < \alpha < 1$. Prove that f is Hölder continuous of order α, i.e. there exists a constant M such that

$$\Big|f(x+h) - f(x)\Big| \leqslant M\,|h|^\alpha$$

for all $x, h \in \mathbb{R}$.

Solution. Clearly $\widehat{f} \in \mathscr{L}^1$, so that we may take f to be a continuous function and hence the claimed pointwise inequality makes sense. Let $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$ be arbitrary. The Fourier inversion formula gives the estimate

$$\begin{aligned}
f(x+h) - f(x) &\ll \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{i(x+h)\xi}\,\mathrm{d}\xi - \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{ix\xi}\,\mathrm{d}\xi \\
&= \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{ix\xi}\left(e^{ih\xi} - 1\right)\mathrm{d}\xi \\
&\ll \int_{-h^{-1}}^{h^{-1}} \Big|\widehat{f}(\xi)\Big| \cdot \Big|e^{ih\xi} - 1\Big|\,\mathrm{d}\xi + \left(\int_{-\infty}^{-h^{-1}} + \int_{h^{-1}}^{\infty}\right) \Big|\widehat{f}(\xi)\Big| \cdot \Big|e^{ih\xi} - 1\Big|\,\mathrm{d}\xi \\
&\ll \Big\|\widehat{f}\Big\|_{\mathscr{L}^\infty} \int_{-h^{-1}}^{h^{-1}} \Big|e^{ih\xi} - 1\Big|\,\mathrm{d}\xi + 4\int_{h^{-1}}^{\infty} \frac{\mathrm{d}\xi}{\xi^{1+\alpha}} \\
&\ll \int_{-h^{-1}}^{h^{-1}} \Big|e^{ih\xi} - 1\Big|\,\mathrm{d}\xi + h^\alpha.
\end{aligned}$$

It suffices to prove that the remaining integral is $O(h^\alpha)$.

If $h \geqslant 1$, then

$$\int_{-h^{-1}}^{h^{-1}} \left|e^{ih\xi} - 1\right| \mathrm{d}\xi \leqslant 2h^{-1} \cdot 2 \leqslant 4 \leqslant 4h^\alpha.$$

On the other hand, if $0 < h < 1$, then

$$\int_{-h^{-1}}^{h^{-1}} \left|e^{ih\xi} - 1\right| \mathrm{d}\xi \leqslant \int_{-1}^{1} \left|h\xi\right| \mathrm{d}\xi \ll h \leqslant h^\alpha,$$

and we are done.

6. Consider the linear partial differential operator $P(x, \partial)$ given by

$$P(x, \partial)\, u(x) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\, \partial^\alpha u(x)\,,$$

where the C^∞-coefficients a_α and all their derivatives are bounded on $\mathbb{R}^n$. Prove that $P(x, \partial)$ maps the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ to itself continuously.

Solution. The claim follows from the observation that for any $u \in \mathscr{S}$ and for all multi-indices α and β we have

$$\begin{aligned}\left\|x^\alpha \partial^\beta P(x, \partial)\, u\right\|_\infty &= \left\|\sum_{|\gamma| \leqslant m} x^\alpha \partial^\beta (a_\gamma \partial^\gamma u)\right\|_\infty \\ &= \left\|\sum_{|\gamma| \leqslant m} x^\alpha \sum_{\delta \leqslant \beta} \binom{\beta}{\delta} \partial^{\beta-\delta} a_\gamma \cdot \partial^{\gamma+\delta} u\right\|_\infty \\ &\leqslant \sum_{|\gamma| \leqslant m} \sum_{\delta \leqslant \beta} \binom{\beta}{\delta} \left\|\partial^{\beta-\delta} a_\gamma\right\|_\infty \cdot \left\|x^\alpha \partial^{\gamma+\delta} u\right\|_\infty.\end{aligned}$$