Introduction to Fourier Analysis
Home assignment 7

1. Let $\varphi(x) = |x|$ for $x \in [0,1]$. Extend φ to $\mathbb{R}$ as a 2-periodic function. Prove that the formula

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$$

defines a continuous function on $\mathbb{R}$.

Solution. Since $|\varphi(x)| \leqslant 1$ for all $x \in \mathbb{R}$, the series defining f may be compared to the absolutely converging geometric series $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$, thereby establishing that the series defining f converges uniformly.

2. We continue to study properties of φ defined in the first exercise. Fix $x_0 \in \mathbb{R}$. For every non-negative integer m define a real number δ_m by the conditions $|\delta_m| = 4^{-m}/2$ and the sign is fixed by the condition that there are no integers between $4^m x_0$ and $4^m (x_0 + \delta_m)$. Consider the quotient

$$\gamma_n = \frac{\varphi\big(4^n(x_0 + \delta_m)\big) - \varphi(4^n x_0)}{\delta_m}.$$

Prove that

a) If $n > m$ then $\gamma_n = 0$.

b) If $0 \leqslant n \leqslant m$, then $|\gamma_n| \leqslant 4^n$ and $|\gamma_m| = 4^m$.

Solution. a) This follows from the fact that φ is 2-periodic and the subsequent observation that when $n > m$, we have

$$\varphi\big(4^n(x + \delta_m)\big) = \varphi\big(4^n x \pm 2^{2(n-m)-1}\big) = \varphi(4^n x)\,.$$

b) For $0 \leqslant n < m$ the inequalities follow directly from the fact that φ is Lipschitz and in fact

$$|\varphi(x) - \varphi(y)| \leqslant |x - y|$$

for all $x, y \in \mathbb{R}$. Finally, the equality $|\gamma_m| = 4^m$ follows from the observation that since the sign of δ_m was chosen in such a way that there are no integers between $4^m (x_0 + \delta_m)$ and $4^m x_0$, the part of the graph of φ between those values looks like a line segment of slope ± 1, and thus

$$|\gamma_m| = \left|\frac{\varphi\big(4^n(x_0 + \delta_m)\big) - \varphi(4^n x_0)}{\delta_m}\right| = \left|\frac{4^m (x_0 + \delta_m) - 4^m x_0}{\delta_m}\right| = 4^m.$$

3. This continues the study of f and φ defined in the previous exercises. Prove that

$$\left|\frac{f(x_0+\delta_m)-f(x_0)}{\delta_m}\right| \geqslant \frac{1}{2}\left(3^m+1\right).$$

Conclude that f is not differentiable at x_0.

Solution. We use the symbols $\gamma_0, \gamma_1, \ldots$ defined in the previous exercise. Since $\left|\gamma_0\right| \leqslant 4^0 = 1$ and $\left|\gamma_m\right| = 4^m$, we have

$$\left|\left(\frac{3}{4}\right)^m \gamma_m + \gamma_0\right| \geqslant \left|\left(\frac{3}{4}\right)^m |\gamma_m| - |\gamma_0|\right| \geqslant 3^m - 1.$$

Similarly, we may estimate

$$\left|\sum_{n=1}^{m-1}\left(\frac{3}{4}\right)^n \gamma_n\right| \leqslant \sum_{n=1}^{m-1}\left(\frac{3}{4}\right)^n |\gamma_n| \leqslant \sum_{n=1}^{m-1} 3^n = 3\cdot\frac{3^{m-1}-1}{3-1} = \frac{3^m-3}{2}.$$

Combining the above estimates with the part a) of the previous exercise gives the inequality

$$\left|\frac{f(x_0+\delta_m)-f(x_0)}{\delta_m}\right| = \left|\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \gamma_n\right| = \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \gamma_n\right|$$

$$\geqslant \left|\left|\left(\frac{3}{4}\right)^m \gamma_m + \gamma_0\right| - \left|\sum_{n=1}^{m-1}\left(\frac{3}{4}\right)^n \gamma_n\right|\right| \geqslant \left|\left|3^m - 1\right| - \left|\frac{3^m-3}{2}\right|\right| = \frac{3^m+1}{2}.$$

Finally, we conclude that the function f can not be differentiable at x_0 since the difference quotient $\frac{f(x_0+h)-f(x_0)}{h}$ can not tend to a finite limit when h tends to zero.

4. Compute the Fourier transform of the characteristic function of the interval $[-a, a]$, where $a > 0$.

Solution. For any $\xi \in \mathbb{R} \setminus \{0\}$, we have

$$\widehat{\chi_{[-a,a]}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{[-a,a]}(x)\, e^{-ix\xi}\, \mathrm{d}x$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ix\xi}\, \mathrm{d}x = \frac{e^{-ia\xi} - e^{ia\xi}}{-\sqrt{2\pi} i\xi} = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin a\xi}{\xi}.$$

Since $\chi_{[-a,a]} \in \mathscr{L}^1(\mathbb{R})$, the Fourier transform $\widehat{\chi_{[-a,a]}}$ is continuous and we must have

$$\widehat{\chi_{[-a,a]}}(0) = \lim_{\xi \longrightarrow 0} \widehat{\chi_{[-a,a]}}(\xi) = \lim_{\xi \longrightarrow 0} \sqrt{\frac{2}{\pi}} \cdot \frac{\sin a\xi}{\xi} = \sqrt{\frac{2}{\pi}} \cdot a.$$

5. Compute the Fourier transform of the function

$$f(x) = \begin{cases} 1 - |x|\,, & |x| \leqslant 1 \\ 0, & |x| > 1. \end{cases}$$

Solution. Again we may compute the Fourier transform directly for any $\xi \in \mathbb{R} \setminus \{0\}$:

$$\begin{aligned}
\widehat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - |x|)\, e^{-ix\xi}\, \mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_{0}^{1} (1 - x) \cos x\xi\, \mathrm{d}x \\
&= \sqrt{\frac{2}{\pi}} \left(\left. \frac{\sin x\xi}{\xi} \right]_{0}^{x=1} - \left. \frac{x \sin x\xi}{\xi} \right]_{0}^{x=1} + \frac{1}{\xi} \int_{0}^{1} \sin x\xi\, \mathrm{d}\xi \right) \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{\sin \xi}{\xi} - \frac{\sin \xi}{\xi} - \left. \frac{\cos x\xi}{\xi^2} \right]_{0}^{x=1} \right) \\
&= \sqrt{\frac{2}{\pi}} \cdot \frac{-\cos \xi + 1}{\xi^2} = \sqrt{\frac{2}{\pi}} \cdot \frac{2 \sin^2 \frac{\xi}{2}}{\xi^2}.
\end{aligned}$$

Since clearly $f \in \mathscr{L}^1(\mathbb{R})$, we obtain $\widehat{f}(0)$ as in the previous exercise:

$$\widehat{f}(0) = \lim_{\xi \longrightarrow 0} \sqrt{\frac{2}{\pi}} \cdot \frac{2 \sin^2 \frac{\xi}{2}}{\xi^2} = \sqrt{\frac{2}{\pi}} \cdot 2 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{\sqrt{2\pi}}.$$