Introduction to Fourier Analysis
Home assignment 6

1. Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^2$ be a differentiable parametrization for the closed curve Γ. Prove that it is a parametrization by the arc length if and only if the length of the curve from $\gamma(a)$ to $\gamma(s)$ for all s is equal to $s-a$, i.e.

$$\int_a^s \left|\gamma'(t)\right| \mathrm{d}t = s - a \quad \text{for all } s \in [a,b].$$

Solution. If γ is a parametrization by the arc length, then the given condition is trivially satisfied. Conversely, if we are given a curve with a parametrization γ satisfying the given condition, then for any $x, y \in [a,b]$, satisfying $x < y$, we have

$$\int_x^y \left|\gamma'(t)\right| \mathrm{d}t = \int_a^y \left|\gamma'(t)\right| \mathrm{d}t - \int_a^x \left|\gamma'(t)\right| \mathrm{d}t = y - a - (x-a) = y - x,$$

thereby showing that γ is a parametrization by the arc length.

2. Prove that any differentiable curve admits a parametrization by the arc length.

Solution. Suppose that we have a continuous curve Γ with a parametrization $\gamma\colon [a,b] \longrightarrow \mathbb{R}^2$ ($a, b \in \mathbb{R}$, $a < b$) with piecewise continuously differentiable[1] coordinate functions and satisfying $|\gamma'(t)| \geqslant \varepsilon > 0$ for all those $t \in [a,b]$ for which the derivative exists.

We consider the arc length function $s\colon [a,b] \longrightarrow \mathbb{R}$ defined by the formula

$$s(t) = \int_a^t \left|\gamma'(\tau)\right| \mathrm{d}\tau$$

for all $\tau \in [a,b]$. This function s is strictly positive and continuous on the interval $[a,b]$. Thus s maps the interval $[a,b]$ bijectively to some other interval

[1]Here we tacitly assume that the derivatives have finite limits from both sides of the non-continuity points. For instance, in our terminology, a function $\alpha\colon [a,b] \longrightarrow \mathbb{R}$ is C^1 if and only if it is continuous and it has a continuous derivative on $]a,b[$ which may be extended to a continuous function on the whole interval $[a,b]$.

$[A, B]$, $(A, B \in \mathbb{R},\ A < B)$. Furthermore, s is continuously differentiable on any interval $]\alpha, \beta[$ $(\alpha, \beta \in [a, b],\ \alpha < \beta)$ on which γ is C^1, and for all $t \in\ ]\alpha, \beta[$,

$$s'(t) = \bigl|\gamma'(t)\bigr|.$$

That is, s is piecewise continuously differentiable on $[a, b]$.

We may now consider the function $r = s^{-1}\colon\ [A, B] \longrightarrow [a, b]$. This is piecewise continuously differentiable since the function s is, and furthermore, for any point t on which r is differentiable,

$$r'(t) = \frac{1}{s'\bigl(r(t)\bigr)} = \frac{1}{\bigl|\gamma'\bigl(r(t)\bigr)\bigr|}.$$

Now the function $\gamma \circ r\colon\ [A, B] \longrightarrow \mathbb{R}^2$ is a new piecewise C^1 parametrization for Γ, and for this parametrization, we have for each $t \in [A, B]$ that

$$\begin{aligned}\int_A^t \bigl|(\gamma \circ r)'(\tau)\bigr| \mathrm{d}\tau &= \int_A^t \bigl|\gamma'\bigl(r(\tau)\bigr)\bigr| \cdot \bigl|r'(\tau)\bigr| \mathrm{d}\tau \\ &= \int_A^t \bigl|\gamma'\bigl(r(\tau)\bigr)\bigr| \cdot \frac{\mathrm{d}\tau}{\bigl|\gamma'\bigl(r(\tau)\bigr)\bigr|} \mathrm{d}\tau = \int_A^t \mathrm{d}\tau = t - A.\end{aligned}$$

We conclude $\gamma \circ r$ is the required arc length parametrization for Γ.

3. Prove the second part of Weyl's criterion: if a sequence $\langle \xi_i \rangle_{i=1}^{\infty}$ is equidistributed then for all $k \in \mathbb{Z} \setminus \bigl\{0\bigr\}$ we have

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \xi_n} \longrightarrow 0 \quad \text{as } N \longrightarrow \infty.$$

Solution. Let $\varepsilon \in \mathbb{R}_+$ be arbitrary. Let $M \in \mathbb{Z}_+$ be so large that $\frac{1}{M} < \frac{\varepsilon}{2}$, and define for each $\ell \in \bigl\{1, 2, \ldots, M\bigr\}$ the interval

$$\mathscr{I}_\ell \stackrel{\text{def}}{=} \left[\frac{\ell - 1}{M}, \frac{\ell}{M}\right[.$$

The equidistribution of the sequence $\langle \xi_n \rangle_{n=1}^{\infty}$ easily implies the equidistribution of the sequence $\langle k\xi_n \rangle_{n=1}^{\infty}$, which in turn implies the existence of a number $N_0 \in \mathbb{Z}_+$ such that for any integer N larger than N_0, we have

$$\left| \frac{\#\bigl(\mathscr{I}_\ell \cap \bigl\{\{k\xi_n\} \bigm| n \in \{1, 2, \ldots, N\}\bigr\}\bigr)}{N} - \frac{1}{M} \right| < \frac{1}{M^2}$$

for each $\ell \in \{1, 2, \ldots, M\}$. Define for each $\ell \in \{1, 2, \ldots, M\}$ and each $N \in \mathbb{Z}_+$ the set of indices

$$\mathscr{A}_{\ell N} \stackrel{\text{def}}{=} \Big\{ n \in \{1, 2, \ldots, N\} \;\Big|\; k\xi_n \in \mathscr{I}_\ell \Big\}.$$

For notational simplicity we use standard notation and write $e(x)$ instead of the clumsier $e^{2\pi i x}$. Having done all the necessary preliminary work, we get for all integers N greater than N_0 and $\frac{4\pi}{\varepsilon}$ that

$$\begin{aligned}
\left|\frac{1}{N}\sum_{n=1}^{N} e(k\xi_n)\right| &= \left|\frac{1}{N}\sum_{\ell=1}^{M}\left(-\frac{1}{M}e\left(\frac{\ell}{M}\right) + \sum_{n\in\mathscr{A}_{\ell N}} e(k\xi_n)\right)\right| \\
\leqslant \frac{1}{N}\sum_{\ell=1}^{M}\left|-\frac{1}{M}e\left(\frac{\ell}{M}\right) + \sum_{n\in\mathscr{A}_{\ell N}} e\left(\frac{\ell}{M}\right)\right| &+ \frac{1}{N}\sum_{\ell=1}^{M}\sum_{n\in\mathscr{A}_{\ell N}}\left|e(k\xi_n) - e\left(\frac{\ell}{M}\right)\right| \\
\leqslant \frac{1}{N}\cdot M\cdot\frac{N}{M^2} + \frac{1}{N}\cdot M\cdot\frac{2\pi}{M} &= \frac{1}{M} + \frac{2\pi}{N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

4. Prove that the sequence $\langle a\log n\rangle_{n=1}^{\infty}$ is not equidistributed for any $a \in \mathbb{R}$.

Solution. For $a = 0$ the claim is obvious whereas for $a \neq 0$ the claim follows from Weyl's criterion together with the observation that

$$\left|\frac{1}{N}\sum_{n=1}^{N} e(a\log n)\right| = \left|\frac{1}{N}\sum_{n=1}^{N}\left(\frac{n}{N}\right)^{2\pi i a}\right| \xrightarrow[N\longrightarrow\infty]{} \left|\int_0^1 x^{2\pi i a}\mathrm{d}x\right| = \frac{1}{|2\pi i a + 1|} \neq 0.$$

5. Suppose that $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ is a continuous function with period 1 and that $\langle \xi_n\rangle_{n=1}^{\infty}$ is an equidistributed sequence in $[0, 1[$. Prove that if in addition

$$\int_0^1 f(x)\,\mathrm{d}x = 0,$$

then uniformly in x,

$$\lim_{N\longrightarrow\infty}\frac{1}{N}\sum_{n=1}^{N} f(x + \xi_n) = 0.$$

Solution. Let us be given an arbitrarily small positive real number ε. Since f is continuous and periodic, we may choose a trigonometric polynomial p such that $\big|f(x) - p(x)\big| < \frac{\varepsilon}{3}$ for all $x \in \mathbb{R}$. Please note that the constant term a_0 of p must have absolute value smaller than $\frac{\varepsilon}{3}$, for otherwise

$$\int_0^1 \big|f(x) - p(x)\big|\mathrm{d}x \geqslant \left|\int_0^1 (f(x) - p(x))\,\mathrm{d}x\right| = \left|\int_0^1 p(x)\,\mathrm{d}x\right| = |a_0| \geqslant \frac{\varepsilon}{3},$$

thereby implying that $\left|f(x)-p(x)\right| \geqslant \frac{\varepsilon}{3}$ for some $x \in \mathbb{R}$. Applying Weyl's criterion for each term of the polynomial $p - a_0$, we conclude that

$$\frac{1}{N}\sum_{n=1}^{N}\bigl(p(x+\xi_n)-a_0\bigr) \xrightarrow[N\longrightarrow\infty]{} 0,$$

and so for sufficiently large integers N, we have

$$\left|\frac{1}{N}\sum_{n=1}^{N}\bigl(p(x+\xi_n)-a_0\bigr)\right| < \frac{\varepsilon}{3}.$$

Finally, we conclude that for sufficiently large integers N we have

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n)\right| \leqslant \frac{1}{N}\sum_{n=1}^{N}\bigl|f(x+\xi_n)-p(x+\xi_n)\bigr| + \frac{1}{N}\left|\sum_{n=1}^{N}\bigl(p(x+\xi_n)-a_0\bigr)\right| + |a_0| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$$

6. Let now $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a bounded measurable function and assume that

$$\int_0^1 f(x)\,\mathrm{d}x = 0.$$

Prove that

$$\lim_{N\longrightarrow\infty}\int_0^1 \left|\frac{1}{N}\sum_{n=1}^{N} f(x+\xi_n)\right|^2 \mathrm{d}x = 0.$$

Solution. Let us be given some arbitrarily small positive real number ε. The basic $\mathscr{L}^2$-theory of Fourier series allows us to choose a trigonometric polynomial p such that

$$\int_0^1 \bigl|f(x)-p(x)\bigr|^2\,\mathrm{d}x < \frac{\varepsilon}{2},$$

and since the integral of f vanishes, we may suppose that the constant term of p vanishes as well.

Using Weyl's criterion termwise to p and its all translates we conclude that

$$\frac{1}{N}\sum_{n=1}^{N} p(x+\xi_n) \xrightarrow[N\longrightarrow\infty]{} 0$$

for all $x \in \mathbb{R}$. Hence the dominated convergence allows us to say that, for sufficiently large integers N, we have

$$\int_0^1 \left| \frac{1}{N} \sum_{n=1}^{N} p(x+\xi_n) \right|^2 \mathrm{d}x < \frac{\varepsilon}{2}.$$

Combining the above considerations leads to the conclusion that, for sufficiently large integers N, we have

$$\begin{aligned} \int_0^1 \left| \frac{1}{N} \sum_{n=1}^{N} f(x+\xi_n) \right|^2 \mathrm{d}x &\leqslant \frac{1}{N} \sum_{n=1}^{N} \int_0^1 \left| f(x+\xi_n) - g(x+\xi_n) \right|^2 \mathrm{d}x \\ &\quad + \int_0^1 \left| \frac{1}{N} \sum_{n=1}^{N} p(x+\xi_n) \right|^2 \mathrm{d}x < \frac{1}{N} \cdot N \cdot \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$