# Introduction to Fourier Analysis
# Home assignment 5

**1.** Assume that $f$ is a $2\pi$-periodic integrable function. Show that for all $n \in \mathbb{Z} \setminus \{0\}$, we have

$$\widehat{f}(n) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(x + \tfrac{\pi}{n}\right) e^{-inx} \, \mathrm{d}x,$$

and hence

$$\widehat{f}(n) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \Big(f(x) - f\left(x + \tfrac{\pi}{n}\right)\Big) e^{-inx} \, \mathrm{d}x.$$

**Solution.** Because of the $2\pi$-periodicity of the integrand in the definition of the Fourier coefficients, we have for any $n \in \mathbb{Z} \setminus \{0\}$ that

$$\begin{aligned}\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-inx} \, \mathrm{d}x &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(x + \tfrac{\pi}{n}\right) e^{-in\left(x + \frac{\pi}{n}\right)} \, \mathrm{d}x \\ &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(x + \tfrac{\pi}{n}\right) e^{-inx} \, \mathrm{d}x.\end{aligned}$$

The second formula is obtained by combining the definition of the Fourier coefficients with the formula just obtained: For all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned}\widehat{f}(n) &= \frac{1}{2} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-inx} \, \mathrm{d}x - \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(x + \tfrac{\pi}{n}\right) e^{-inx} \, \mathrm{d}x \right) \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \Big(f(x) - f\left(x + \tfrac{\pi}{n}\right)\Big) e^{-inx} \, \mathrm{d}x.\end{aligned}$$

**2.** Assume that the function $f$ above also satisfies the *Hölder condition with exponent* $\alpha$,

$$\left|f(x+h) - f(x)\right| \leqslant C|h|^{\alpha},$$

for some $0 < \alpha \leqslant 1$ and all real $x$ and $h$. Show that the Fourier coefficients of $f$ satisfy

$$\widehat{f}(n) = O\left(|n|^{-\alpha}\right).$$

**Solution.** This follows directly from the second conclusion of the previous exercise: For any $n \in \mathbb{Z} \setminus \{0\}$, we have

$$\begin{aligned}\left|\widehat{f}(n)\right| &= \left|\frac{1}{4\pi}\int_{-\pi}^{\pi}\left(f(x) - f\left(x + \tfrac{\pi}{n}\right)\right) e^{-inx}\,\mathrm{d}x\right| \\ &\leqslant \frac{1}{4\pi}\int_{-\pi}^{\pi}\left| f(x) - f\left(x + \tfrac{\pi}{n}\right)\right| \mathrm{d}x \leqslant \frac{1}{4\pi}\int_{-\pi}^{\pi} C\left|\frac{\pi}{n}\right|^{\alpha} \mathrm{d}x = \frac{C\pi^{\alpha}}{2\,|n|^{\alpha}}.\end{aligned}$$

**3.** Prove that the result given in the previous exercise is sharp by showing that the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x},$$

where $0 < \alpha < 1$, satisfies the Hölder condition with exponent $\alpha$ and that

$$\widehat{f}(N) = N^{-\alpha},$$

when $N = 2^k$.

**Solution.** We can easily compare the series defining the function $f$ to a geometric series to see that it converges absolutely everywhere. We conclude that the series defining the function $f$ is the Fourier series of $f$, the series converges pointwise everywhere, and due to the uniqueness property of Fourier series, we have $\widehat{f}\left(2^k\right) = 2^{-k\alpha}$ for all non-negative integers $k$.

To see the Hölder continuity of $f$, we let $x$ and $h$ be arbitrary real numbers. We first observe that it suffices to prove the Hölder continuity only in the case $h \in ]0, 2\pi[$, since if that case has already been taken care of, the $2\pi$-periodicity of $f$ implies that for any $n \in \mathbb{Z}_+$ and any $h \in [0, 2\pi[$, we have also

$$\left|f(x + h + 2\pi n) - f(x)\right| = \left|f(x+h) - f(x)\right| \ll_{\alpha} \left|h\right|^{\alpha} \leqslant \left|h + 2\pi n\right|^{\alpha}.$$

The proof of Hölder continuity will require the following simple observations. For any $x \in \mathbb{R}$, we trivially have $\left|e^{ix} - 1\right| \leqslant 2$, and it is a simple geometrical observation that for $x \in [-1, 1]$, we have $\left|e^{ix} - 1\right| \leqslant \left|x\right|$. Now, for any $x \in \mathbb{R}$ and any $h \in ]0, 2\pi[$, we have

$$\begin{aligned}\left|f(x+h) - f(x)\right| &= \left|\sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k(x+h)} - \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x}\right| \\ \leqslant \sum_{k=0}^{\infty} 2^{-k\alpha}\left|e^{i2^k h} - 1\right| &= \sum_{\substack{k \geqslant 0 \\ 1 \leqslant 2^k \leqslant |h|^{-1}}} 2^{-k\alpha}\left|e^{i2^k h} - 1\right| + \sum_{\substack{k \geqslant 0 \\ 2^k > |h|^{-1}}} 2^{-k\alpha}\left|e^{i2^k h} - 1\right|\end{aligned}$$

$$\leqslant \sum_{\substack{k\geqslant 0\\ 1\leqslant 2^k\leqslant |h|^{-1}}} 2^{-k\alpha}\cdot 2^k\big|h\big| + \sum_{\substack{k\geqslant 0\\ 2^k>|h|^{-1}}} 2^{-k\alpha}\cdot 2 = \big|h\big| \sum_{\substack{k\geqslant 0\\ 1\leqslant 2^k\leqslant |h|^{-1}}} 2^{k(1-\alpha)} + 2\sum_{\substack{k\geqslant 0\\ 2^k>|h|^{-1}}} \big(2^{-\alpha}\big)^k$$

$$\leqslant \big|h\big|\cdot \frac{2^{(1-\alpha)\left(1+\mathrm{lb}\,|h|^{-1}\right)}-1}{2^{1-\alpha}-1} + 2\cdot\frac{\big(2^{-\alpha}\big)^{\mathrm{lb}\,|h|^{-1}}}{1-2^{-\alpha}}$$

$$= \frac{\Big(2^{1-\alpha}\big|h\big|^{\alpha-1}-1\Big)\big|h\big|}{2^{1-\alpha}-1} + \frac{2\big|h\big|^{\alpha}}{1-2^{-\alpha}} \leqslant \left(\frac{2^{1-\alpha}}{2^{1-\alpha}-1}+\frac{2}{1-2^{-\alpha}}\right)\big|h\big|^{\alpha},$$

where lb signifies the logarithm in base two.

**4.** Assume that $f$ is a $2\pi$-periodic function that satisfies a Lipschitz condition with constant $K$, i.e.

$$\big|f(x)-f(y)\big| \leqslant K\big|x-y\big| \qquad \text{for all } x \text{ and } y.$$

Define for $h>0$

$$g_h(x) = f(x+h) - f(x-h)\,.$$

Prove that

$$\frac{1}{2\pi}\int_0^{2\pi}\big|g_h(x)\big|^2\mathrm{d}x = \sum_{n=-\infty}^{\infty} 4\big|\sin nh\big|^2\big|\widehat{f}(n)\big|^2,$$

and show that

$$\sum_{n=-\infty}^{\infty}\big|\sin nh\big|^2\big|\widehat{f}(n)\big|^2 \leqslant K^2h^2.$$

**Solution.** The Lipschitz condition implies that $f$ and $g$ are continuous on the interval $[-\pi,\pi]$ and hence it is clear that they are periodic $\mathscr{L}^2$-functions. The proposed equality is an immediate consequence of Parseval's formula and the observation that for all $n\in\mathbb{Z}$, we have

$$\begin{aligned}\widehat{g_h}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(f(x+h)-f(x-h)\big)e^{-inx}\,\mathrm{d}x\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-in(x-h)}\,\mathrm{d}x - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-in(x+h)}\,\mathrm{d}x\\ &= \widehat{f}(n)\,e^{inh} - \widehat{f}(n)\,e^{-inh} = 2i\widehat{f}(n)\sin nh.\end{aligned}$$

The inequality follows easily:

$$4\sum_{n=-\infty}^{\infty}\sin^2 nh\,\big|\widehat{f}(n)\big|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\big|g_h(x)\big|^2\mathrm{d}x$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\big|f(x+h)-f(x-h)\big|^2\mathrm{d}x \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\big|K\cdot 2h\big|^2\mathrm{d}x = 4K^2\big|h\big|^2.$$

**5.** Let $p$ be a positive integer and $f$ as in the previous exercise. By choosing $h=\frac{\pi}{2^{p+1}}$ above show that

$$\sum_{2^{p-1}<|n|\leqslant 2^p}\big|\widehat{f}(n)\big|^2 \leqslant \frac{K^2\pi^2}{2^{2p+1}}.$$

**Solution.** We first observe that $\big|\sin x\big| \geqslant \frac{1}{\sqrt{2}}$ for all $x\in\big[-\frac{\pi}{2},-\frac{\pi}{4}\big]\cup\big[\frac{\pi}{4},\frac{\pi}{2}\big]$. Then we obtain from the inequality which was proved in the previous exercise that

$$\frac{K^2\pi^2}{2^{2p+2}} = K^2\left|\frac{\pi}{2^{p+1}}\right|^2 \geqslant \sum_{n=-\infty}^{\infty}\sin^2\frac{n\pi}{2^{p+1}}\big|\widehat{f}(n)\big|^2 \geqslant \sum_{2^{p-1}<|n|\leqslant 2^p}\sin^2\frac{n\pi}{2^{p+1}}\big|\widehat{f}(n)\big|^2$$
$$\geqslant \sum_{2^{p-1}<|n|\leqslant 2^p}\left(\frac{1}{\sqrt{2}}\right)^2\big|\widehat{f}(n)\big|^2 = \frac{1}{2}\sum_{2^{p-1}<|n|\leqslant 2^p}\big|\widehat{f}(n)\big|^2.$$

**6.** Let again $f$ be as above. By estimating the sum $\sum_{2^{p-1}<|n|\leqslant 2^p}\big|\widehat{f}(n)\big|$ prove that the Fourier series of $f$ converges absolutely.

**Solution.** Using the inequality from the previous exercise and the Cauchy–Schwarz–Bunyakovsky inequality, we get

$$\sum_{2^{p-1}<|n|\leqslant 2^p}\big|\widehat{f}(n)\big| \leqslant \sqrt{\sum_{2^{p-1}<|n|\leqslant 2^p}1^2}\sqrt{\sum_{2^{p-1}<|n|\leqslant 2^p}\big|\widehat{f}(n)\big|^2} \leqslant \sqrt{2^p}\sqrt{\frac{K^2\pi^2}{2^{2p+1}}} = \frac{K\pi}{2^{\frac{p+1}{2}}}.$$

Therefore

$$\sum_{|n|>1}\big|\widehat{f}(n)\big| = \sum_{p=1}^{\infty}\sum_{2^{p-1}<|n|\leqslant 2^p}\big|\widehat{f}(n)\big| \leqslant K\pi\sum_{p=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^{p+1} = \frac{K\pi}{2\big(1-\frac{1}{\sqrt{2}}\big)} < \infty.$$