Introduction to Fourier Analysis
Home assignment 4

1. Let $f\colon [-\pi,\pi] \longrightarrow \mathbb{R}$, $f(\vartheta) = \left|\vartheta\right|$. Use Parseval's formula to compute the sums

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Solution. In the fifth exercise of the first exercise set we proved that the Fourier coefficients of f are given by the formulas

$$\widehat{f}(n) = \begin{cases} \frac{\pi}{2} & \Longleftarrow n = 0, \\ 0 & \Longleftarrow n \neq 0 \text{ and } 2 \mid n, \\ \frac{-2}{\pi n^2} & \Longleftarrow 2 \nmid n, \end{cases}$$

for all $n \in \mathbb{Z}$. Parseval's formula now says that

$$\frac{\pi^2}{3} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\vartheta\right|^2 \mathrm{d}\vartheta = \sum_{n=-\infty}^{\infty} \left|\widehat{f}(n)\right|^2 = \frac{\pi^2}{4} + 2\sum_{\substack{n \geqslant 1 \\ 2 \nmid n}} \left(-\frac{2}{\pi n^2}\right)^2 = \frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{\substack{n \geqslant 1 \\ 2 \nmid n}} \frac{1}{n^4},$$

and thus

$$\sum_{\substack{n \geqslant 1 \\ 2 \nmid n}} \frac{1}{n^4} = \frac{\pi^2}{8}\left(\frac{\pi^2}{3} - \frac{\pi^2}{4}\right) = \frac{\pi^2}{8} \cdot \frac{\pi^2}{12} = \frac{\pi^4}{96}.$$

As in the exercise 1.6, it is easy to obtain the other series. Since

$$\left(1 - \frac{1}{16}\right)\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^4} - \sum_{n=1}^{\infty} \frac{1}{(2n)^4} = \sum_{\substack{n \geqslant 1 \\ 2 \nmid n}} \frac{1}{n^4} = \frac{\pi^4}{96},$$

we must have

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{16}{15} \cdot \frac{\pi^4}{96} = \frac{\pi^4}{90}.$$

2. Let f be the 2π-periodic odd function defined on $[0,\pi]$ by $f(\vartheta) = \vartheta\,(\pi - \vartheta)$. Prove that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Solution. We first compute the square of the $\mathscr{L}^2$-norm of the function f in the interval $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} \left|f(\vartheta)\right|^2 \mathrm{d}\vartheta = 2\int_{0}^{\pi} \vartheta^2 \left(\pi - \vartheta\right)^2 \mathrm{d}\vartheta = 2\int_{0}^{\pi} \left(\pi^2\vartheta^2 - 2\pi\vartheta^3 + \vartheta^4\right) \mathrm{d}\vartheta$$

$$= 2\left(\pi^2 \cdot \frac{\vartheta^3}{3} - 2\pi \cdot \frac{\vartheta^4}{4} + \frac{\vartheta^5}{5}\right)\Bigg]_0^{\vartheta=\pi} = \frac{2\pi^5}{3} - \pi^5 + \frac{2\pi^5}{5} = \frac{\pi^5}{15}.$$

We recall that in the fourth exercise of the first exercise set we proved for all $n \in \mathbb{Z}$ that the n^{th} Fourier coefficient of f vanishes if n is even and is equal to $-\frac{4i}{\pi n^3}$ if n is odd. By Parseval's formula, we have

$$\frac{\pi^4}{30} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(\vartheta)\right|^2 \mathrm{d}\vartheta = \sum_{n=-\infty}^{\infty} \left|\widehat{f}(n)\right|^2 = 2\sum_{\substack{n \geqslant 1 \\ 2 \nmid n}} \left|-\frac{4i}{\pi n^3}\right|^2 = \frac{32}{\pi^2}\sum_{\substack{n \geqslant 1 \\ 2 \nmid n}} \frac{1}{n^6},$$

and thus

$$\sum_{\substack{n \geqslant 1 \\ 2 \nmid n}} \frac{1}{n^6} = \frac{\pi^2}{32} \cdot \frac{\pi^4}{30} = \frac{\pi^6}{960}.$$

In the same way as before we reason that

$$\left(1 - \frac{1}{64}\right)\sum_{n=1}^{\infty} \frac{1}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^6} - \sum_{n=1}^{\infty} \frac{1}{(2n)^6} = \sum_{\substack{n \geqslant 1 \\ 2 \nmid n}} \frac{1}{n^6} = \frac{\pi^6}{960},$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{64}{63} \cdot \frac{\pi^6}{960} = \frac{\pi^6}{945}.$$

3. Show that for α not an integer, the Fourier series of

$$\frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha}, \qquad x \in [0, 2\pi],$$

is given by

$$\sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n + \alpha}.$$

Solution. This is just a straightforward computation: Let $n \in \mathbb{Z}$. Then the n^{th} Fourier coefficient of the given function must be

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\frac{\pi}{\sin\pi\alpha}e^{i(\pi-x)\alpha}e^{-inx}\mathrm{d}x &= \frac{e^{i\pi\alpha}}{2\sin\pi\alpha}\int_0^{2\pi}e^{-i(n+\alpha)x}\mathrm{d}x \\ &= \frac{e^{i\pi\alpha}}{2\sin\pi\alpha}\cdot\left.\frac{e^{-i(n+\alpha)x}}{-i\,(n+\alpha)}\right]_0^{2\pi} = \frac{e^{i\pi\alpha}\left(e^{-2i\pi\alpha}-1\right)}{-2i\,(n+\alpha)\sin\pi\alpha} \\ &= \frac{e^{i\pi\alpha}-e^{-i\pi\alpha}}{2i}\cdot\frac{1}{(n+\alpha)\sin\pi\alpha} = \frac{1}{n+\alpha},\end{aligned}$$

as required.

4. Assume that the complex sequence $\langle a_n\rangle$ has the property that

$$\sum\left|a_n b_n\right| < \infty$$

for all complex sequences $\langle b_n\rangle$ such that $\sum\left|b_n\right|^2 < \infty$. Prove that $\sum\left|a_n\right|^2 < \infty$.

Solution. We shall prove the claim in contrapositive form. Let $\langle a_n\rangle_{n=1}^{\infty}$ be an arbitrary sequence of complex numbers such that $\sum_{n=1}^{\infty}\left|a_n\right|^2 = \infty$. We first suppose that the sequence $\langle a_n\rangle_{n=1}^{\infty}$ is bounded, i.e. that $\left|a_n\right| \leqslant M$ for all $n \in \mathbb{Z}_+$ for some $M \in \mathbb{R}_+$. Then we may find disjoint subsets E_1, E_2, … of $\mathbb{Z}_+$ such that

$$\sum_{n\in E_k}\left|a_n\right|^2 \in [M, 2M[$$

for all $k \in \mathbb{Z}_+$. We define a sequence $\langle b_n\rangle_{n=1}^{\infty}$ of complex numbers by defining for all $n \in \mathbb{Z}_+$ the number b_n to be $\frac{1}{k}a_n$, if $n \in E_k$, for some $k \in \mathbb{Z}_+$. Otherwise we let $b_n = 0$. Then the sequence $\langle b_n\rangle_{n=1}^{\infty}$ is in ℓ^2 since

$$\sum_{n=1}^{\infty}\left|b_n\right|^2 = \sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{n\in E_k}\left|a_n\right|^2 < 2M\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty,$$

and yet

$$\sum_{n=1}^{\infty}\left|a_n b_n\right| = \sum_{k=1}^{\infty}\frac{1}{k}\sum_{n\in E_k}\left|a_n\right|^2 \geqslant M\sum_{k=1}^{\infty}\frac{1}{k} = \infty.$$

Next we assume that the sequence $\langle a_n\rangle_{n=1}^{\infty}$ is unbounded. Let $\langle a_{n_k}\rangle_{k=1}^{\infty}$ be a subsequence of $\langle a_n\rangle_{n=1}^{\infty}$ such that $\left|a_{k_1}\right| < \left|a_{k_2}\right| < \left|a_{k_3}\right| < \ldots$ We define a sequence of complex numbers $\langle b_n\rangle_{n=1}^{\infty}$ by choosing $b_n = \frac{1}{k}$, if $n = n_k$ for some $k \in \mathbb{Z}_+$, and $b_n = 0$ otherwise. Then

$$\sum_{n=1}^{\infty} \left|b_n\right|^2 = \sum_{k=1}^{\infty} \left|b_{n_k}\right|^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

and

$$\sum_{n=1}^{\infty} \left|a_n b_n\right| = \sum_{k=1}^{\infty} \left|a_{n_k} b_{n_k}\right| \geqslant \left|a_{n_1}\right| \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

5. Let X be a normed space and Y a Banach space. Let $\langle\Lambda_n\rangle$ be a sequence of bounded linear maps $X \longrightarrow Y$ which are uniformly bounded, i.e. there exists a constant $M < \infty$ such that for all n we have $\left\|\Lambda_n\right\| \leqslant M$. Assume that there exists a dense set $E \subset X$ so that $\langle\Lambda_n x\rangle$ converges in Y for all $x \in E$. Prove that $\langle\Lambda_n x\rangle$ converges in Y for all $x \in X$.

Solution. Let $x \in X$ be arbitrary and let us be given some number $\varepsilon \in \mathbb{R}_+$. We choose an element y from the set E so that $\left\|x - y\right\| < \frac{\varepsilon}{3(M+1)}$. Since the sequence $\langle\Lambda_n y\rangle_{n=1}^{\infty}$ converges, we have $\left\|\Lambda_n y - \Lambda_m y\right\| < \frac{\varepsilon}{3}$ for sufficiently large positive integers m and n. Thus, for sufficiently large positive integers m and n,

$$\begin{aligned} \left\|\Lambda_n x - \Lambda_m x\right\| &\leqslant \left\|\Lambda_n x - \Lambda_n y\right\| + \left\|\Lambda_n y - \Lambda_m y\right\| + \left\|\Lambda_m y - \Lambda_m x\right\| \\ &\leqslant 2M\left\|x - y\right\| + \left\|\Lambda_n y - \Lambda_m y\right\| < 2M \cdot \frac{\varepsilon}{3\,(M+1)} + \frac{\varepsilon}{3} < \varepsilon. \end{aligned}$$

This proves that $\langle\Lambda_n x\rangle_{n=1}^{\infty}$ is a fundamental sequence in Y for all x, and since Y is a Banach space, we are done.

Remark. The original problem text contained a misprint; it said that X was to be a Banach space and Y an arbitrary normed space. However, with these assumptions the claim does not hold. A simple counterexample is furnished by the case in which $X = \ell^2$,

$$Y = c_{00} = \left\{\langle x_n\rangle_{n=1}^{\infty} \in \ell^2 \;\middle|\; \exists N \in \mathbb{Z}_+ \colon 0 = x_N = x_{N+1} = x_{N+2} = \ldots\right\},$$

the space Y is provided with the ℓ^2-norm, and for each $N \in \mathbb{Z}_+$,

$$\Lambda_N = \langle x_n\rangle_{n=1}^{\infty} \longmapsto \langle x_1, x_2, \ldots, x_N, 0, 0, \ldots\rangle : \ell^2 \longrightarrow c_{00}.$$

Then clearly the sequence $\langle\Lambda_N x\rangle_{N=1}^{\infty}$ becomes eventually stationary for each $x \in c_{00}$, and of course c_{00} is dense in ℓ^2. However, it is also clear that the sequence $\langle\Lambda_N x\rangle_{N=1}^{\infty}$ can not converge in c_{00} for any $x \in \ell^2 \setminus c_{00}$.

6. Let $f \in \mathscr{C}(\mathbb{T})$. Prove that

$$\lim_{N \longrightarrow \infty} \frac{S_N f(\vartheta)}{\log N} = 0$$

uniformly. On the other hand, assume that the complex sequence $\langle \lambda_n \rangle$, $n \in \mathbb{N}$, satisfies

$$\lim_{n \longrightarrow \infty} \frac{\lambda_n}{\log n} = 0.$$

Prove that there exists an $f \in \mathscr{C}(\mathbb{T})$ such that the sequence $\left\langle \frac{S_N f(0)}{\lambda_N} \right\rangle$ is unbounded.

Solution. Let us consider the operators $S_N = f \longmapsto S_N f \colon \mathscr{C}(\mathbb{T}) \longrightarrow \mathscr{C}(\mathbb{T})$, $N \in \mathbb{Z}_+$. It is clear that the norm of the operator Λ_N is at most $\frac{1}{2\pi}$ times the $\mathscr{L}^1$-norm of the N^{th} Dirichlet kernel D_N. By slightly altering the solution to the sixth exercise of the second problem set, we see that for sufficiently large $N \in \mathbb{Z}_+$, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| D_N \right| \mathrm{d}x = \frac{1}{\pi} \int_{0}^{\pi} \left| \frac{\sin\left(N + \frac{1}{2}\right) x}{\sin \frac{x}{2}} \right| \mathrm{d}x$$

$$\ll \int_0^{\pi} \frac{\left|\sin\left(N+\frac{1}{2}\right)x\right| \mathrm{d}x}{x} = \int_0^{\pi\left(N+\frac{1}{2}\right)} \frac{\left|\sin y\right| \mathrm{d}y}{y} \leqslant \int_0^{\pi(N+1)} \frac{\left|\sin y\right| \mathrm{d}y}{y}$$

$$\leqslant \int_0^{\pi} \frac{\left|\sin y\right| \mathrm{d}y}{y} + \sum_{n=1}^{N} \frac{1}{\pi n} \int_{\pi n}^{\pi(n+1)} \left|\sin y\right| \mathrm{d}y \ll 1 + \sum_{n=1}^{N} \frac{1}{n} \ll \log N.$$

The purpose of this little detour is to see that the sequence of operators $\left\langle \frac{S_N}{\log N} \right\rangle_{N=N_0}^{\infty}$ is uniformly bounded by some fixed number $M \in \mathbb{R}_+$ for some $N_0 \in \mathbb{Z}_+$. Let $f \in \mathscr{C}(\mathbb{T})$ be arbitrary, and let us be given some arbitrary $\varepsilon \in \mathbb{R}_+$. Since the subspace of trigonometric polynomials is dense in $\mathscr{C}(\mathbb{T})$, we may choose a trigonometric polynomial p so that $\left\| f - p \right\| < \frac{\varepsilon}{2M}$. For sufficiently large $N \in \mathbb{Z}_+$, we also have

$$\left\| \frac{S_N p}{\log N} \right\| < \frac{\varepsilon}{2},$$

because the sequence $\langle S_N p \rangle_{N=1}^{\infty}$ is eventually stationary. But now for sufficiently large $N \in \mathbb{Z}_+$, we have

$$\left\| \frac{S_N f}{\log N} \right\| \leqslant \left\| \frac{S_N (f-p)}{\log N} \right\| + \left\| \frac{S_N p}{\log N} \right\| \leqslant \frac{M}{\log N} \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2} < \varepsilon.$$

We conclude that

$$\lim_{N \longrightarrow \infty} \frac{S_N f}{\log N} = 0$$

in $\mathscr{C}(\mathbb{T})$ for every $f \in \mathscr{C}(\mathbb{T})$, as was to be proved.

For the second part of the exercise, we consider the sequence $\langle \Lambda_N \rangle_{N=1}^{\infty}$ of linear functionals defined as follows:

$$\Lambda_N = f \longmapsto S_N f(0) : \mathscr{C}(\mathbb{T}) \longrightarrow \mathbb{C}.$$

It was proven in the lectures that

$$\|\Lambda_N\| \gg \log N, \qquad (N \longrightarrow \infty)$$

and therefore

$$\left\| \frac{\Lambda_N}{\lambda_N} \right\| \longrightarrow \infty. \qquad (N \longrightarrow \infty)$$

Thus the sequence of operators $\left\langle \frac{\Lambda_N}{\lambda_N} \right\rangle_{N=1}^{\infty}$ is not uniformly bounded and the Banach–Steinhaus theorem immediately guarantees the existence of a function $f \in \mathscr{C}(\mathbb{T})$ such that the sequence $\left\langle \frac{\Lambda_N f}{\lambda_N} \right\rangle_{N=1}^{\infty}$ is unbounded.