Introduction to Fourier Analysis
Home assignment 3

1. Show that the Fejér kernel can be written as

$$F_N(x) = \frac{1}{N} \cdot \frac{\sin^2 \frac{Nx}{2}}{\sin^2 \frac{x}{2}}.$$

Solution. This only requires a direct computation, the details of which are perhaps most easily revealed by working from the both ends of the proposed identity. For any $x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, we have

$$\begin{aligned}
F_N(x) &= \frac{1}{N} \sum_{k=0}^{N-1} D_k(x) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{\sin\left(k+\frac{1}{2}\right)x}{\sin \frac{x}{2}} \\
&= \frac{1}{N \sin \frac{x}{2}} \sum_{k=0}^{N-1} \frac{e^{i\left(k+\frac{1}{2}\right)x} - e^{-i\left(k+\frac{1}{2}\right)x}}{2i} \\
&= \frac{1}{N \sin \frac{x}{2}} \cdot \frac{e^{i\frac{x}{2}} \cdot \frac{e^{iNx}-1}{e^{ix}-1} - e^{-i\frac{x}{2}} \cdot \frac{e^{-iNx}-1}{e^{-ix}-1}}{2i} \\
&= \frac{1}{N \sin \frac{x}{2}} \cdot \frac{e^{iNx} - 1 + e^{-iNx} - 1}{2i\left(e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}\right)} = \frac{1}{N \sin \frac{x}{2}} \cdot \frac{e^{iNx} - 2 + e^{-iNx}}{2i\left(e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}\right)} \\
&= \frac{1}{N \sin \frac{x}{2}} \cdot \frac{\left(e^{i\frac{Nx}{2}} - e^{-i\frac{Nx}{2}}\right)^2}{2i\left(e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}\right)} = \frac{1}{N} \cdot \frac{\sin^2 \frac{Nx}{2}}{\sin^2 \frac{x}{2}}
\end{aligned}$$

2. Prove that the series $\sum_{k=0}^{\infty} (-1)^k (1+k)$ is not Cesàro summable.

Solution. So, we need to consider the divergent series

$$1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 + 11 - 12 + \dots$$

The sequence of partial sums of this series is easily seen to be

$$1, \quad -1, \quad 2, \quad -2, \quad 3, \quad -3, \quad 4, \quad -4, \quad 5, \quad -5, \quad 6, \quad -6, \quad \dots$$

Therefore the corresponding sequence of Cesàro means must be

$$1, \quad 0, \quad \frac{2}{3}, \quad 0, \quad \frac{3}{5}, \quad 0, \quad \frac{4}{7}, \quad 0, \quad \frac{5}{9}, \quad 0, \quad \frac{6}{11}, \quad 0, \dots$$

Here the even terms vanish and the sequence of the odd terms converges to $\frac{1}{2}$. Hence the series under consideration is not Cesàro summable.

3. Prove that if the series $\sum c_n$ of complex numbers is Cesàro summable, and the sum is σ, then $\sum c_n$ is Abel summable to σ.

Solution. We are provided with a Cesàro summable series of complex numbers $\sum_{n=1}^{\infty} c_n$ having the Cesàro sum σ. Let $\langle s_n \rangle_{n=1}^{\infty}$ be the sequence of partial sums of the series under consideration, and let $\langle \sigma_n \rangle_{n=1}^{\infty}$ be the corresponding sequence of Cesàro means. Let us assume first that $\sigma = 0$, and suppose that we have been provided with an arbitrarily small $\varepsilon \in \mathbb{R}_+$. Then there must exist a fixed positive integer N_ε such that $|\sigma_n| < \varepsilon$ for all integers n greater than N_ε.

We begin by taking a look at the series $\sum_{n=1}^{\infty} n\sigma_n r^n$ for some fixed value of $r \in [0, 1[$. This series is always absolutely convergent, since

$$\begin{aligned}\sum_{n=1}^{\infty} |n\sigma_n r^n| &\leqslant \sum_{n=1}^{N_\varepsilon} |n\sigma_n r^n| + \sum_{n=N_\varepsilon+1}^{\infty} n\,|\sigma_n|\, r^n \leqslant C_\varepsilon + \varepsilon \sum_{n=N_\varepsilon+1}^{\infty} nr^n \\ &= C_\varepsilon + \varepsilon \cdot \frac{(N_\varepsilon + 1)\, r^{N_\varepsilon+1}}{1-r} + \frac{\varepsilon r^{N_\varepsilon+2}}{(1-r)^2} \leqslant C_\varepsilon + \frac{\varepsilon\,(N_\varepsilon+1)}{1-r} + \frac{\varepsilon}{(1-r)^2},\end{aligned}$$

where the constant C_ε is, of course, just the sum with N_ε terms.

Two summations by parts say that for any sufficiently large $N \in \mathbb{Z}_+$ and any $r \in [0, 1[$, we have

$$\begin{aligned}\sum_{n=1}^{N} c_n r^n &= s_N r^N - \sum_{n=1}^{N-1} \left(r^{n+1} - r^n\right) s_n \\ &= s_N r^N - (N-1)\,\sigma_{N-1}\left(r^{N+1} - r^N\right) + \sum_{n=1}^{N-2} n\sigma_n \left(r^{n+2} - 2r^{n+1} + r^n\right).\end{aligned}$$

Letting here $N \longrightarrow \infty$ gives

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n,$$

and consequently

$$\left|\sum_{n=1}^{\infty} c_n r^n\right| \leqslant C_\varepsilon\,(1-r)^2 + \varepsilon\,(N_\varepsilon+1)\,(1-r) + \varepsilon,$$

and here the right-hand side can be pushed as near to ε as one wishes to simply by taking values of r to be sufficiently close to 1. That is, the series $\sum_{n=1}^{\infty} c_n$ is Abel summable to zero.

We still need to consider the general case $\sigma \in \mathbb{C}$. In this case the series $-\sigma + c_1 + c_2 + \ldots$ must be Cesàro summable to zero since its N^{th} Cesàro mean is

$$-\sigma + \frac{1}{N}\sum_{n=1}^{N-1} s_n = -\sigma + \frac{N-1}{N}\sigma_{N-1},$$

for arbitrary $N \in \mathbb{Z}_+$ and the last expression tends to zero as $N \longrightarrow \infty$. Hence the Abel series $-\sigma r + \sum_{n=1}^{\infty} c_n r^{n+1}$ converges absolutely for all $r \in [0,1[$ and

$$-\sigma r + \sum_{n=1}^{\infty} c_n r^{n+1} \xrightarrow[r \longrightarrow 1-]{} 0,$$

thereby implying that the series $\sum_{n=1}^{\infty} c_n r^n$ also converges absolutely for all $r \in [0,1[$, and

$$\sum_{n=1}^{\infty} c_n r^n \xrightarrow[r \longrightarrow 1-]{} \sigma.$$

4. Under certain conditions one can reverse the summability results, i.e. from Abel or Cesàro summability deduce the summability of the original series. These kinds of theorems are known as *Tauberian theorems*. As an example, assume that the sequence $\langle c_n \rangle$ of complex numbers satisfies $nc_n \longrightarrow 0$ as $n \longrightarrow \infty$, and that it is Cesàro summable to σ. Prove that $\sum c_n = \sigma$.

Solution. Let $\langle s_n \rangle_{n=1}^{\infty}$ be the sequence of partial sums of the Cesàro summable series $\sum_{n=1}^{\infty} c_n$, and let $\langle \sigma_n \rangle_{n=1}^{\infty}$ be the corresponding sequence of Cesàro means. We know that $nc_n \longrightarrow 0$ and $\sigma_n \longrightarrow \sigma$ as $n \longrightarrow \infty$ and we are supposed to prove that $s_n \longrightarrow \sigma$ as $n \longrightarrow \infty$.

We may write the partial sum s_N in the form

$$\sum_{n=1}^{N} c_n = \frac{1}{N-1}\sum_{n=1}^{N-1}(N-n)\,c_n + \frac{1}{N-1}\sum_{n=1}^{N-1} nc_{n+1} = \sigma_{N-1} + \frac{1}{N-1}\sum_{n=1}^{N-1} nc_{n+1}.$$

The first term of the right-most sum converges to σ as $N \longrightarrow \infty$ by assumption, and therefore we only need to show that the last sum tends to zero as $N \longrightarrow \infty$.

Let us be given an arbitrarily small $\varepsilon \in \mathbb{R}_+$, and let $N_0 \in \mathbb{Z}_+$ be such that

$$\left|nc_n\right| < \frac{\varepsilon}{2},$$

for all integers n greater than N_0. Now, for sufficiently large $N \in \mathbb{Z}_+$, we must have

$$\left|\frac{1}{N-1}\sum_{n=1}^{N_0} nc_{n+1}\right| < \frac{\varepsilon}{2},$$

and consequently

$$\left|\frac{1}{N-1}\sum_{n=1}^{N-1} nc_{n+1}\right| \leqslant \left|\frac{1}{N-1}\sum_{n=1}^{N_0} nc_{n+1}\right| + \frac{1}{N-1}\sum_{n=N_0+1}^{N-1}\left|nc_{n+1}\right|$$
$$< \frac{\varepsilon}{2} + \frac{N-N_0}{N-1}\cdot\frac{\varepsilon}{2} \leqslant \varepsilon.$$

5. Again, assume that the sequence $\langle c_n\rangle$ of complex numbers satisfies $nc_n \to 0$ as $n \longrightarrow \infty$, but now that it is Abel summable to σ. Prove that $\sum c_n = \sigma$.

Solution. Let us be given some arbitrarily small $\varepsilon \in \mathbb{R}_+$. For any $N \in \mathbb{Z}_+$, denote $r_N = 1 - \frac{1}{N}$. Then clearly $r_N \longrightarrow 1-$ as $N \longrightarrow \infty$, and so we may choose N to be so large that

$$\left|\sum_{n=1}^{\infty} c_n r_N^n - \sigma\right| < \frac{\varepsilon}{3}.$$

Since $nc_n \longrightarrow 0$ as $n \longrightarrow \infty$, there exists a positive integer N_0 such that $\left|nc_n\right| < \frac{\varepsilon}{6}$ for all integers $n > N_0$. Then we have

$$\left|\sum_{n=1}^{N} c_n\left(1 - r_N^n\right)\right| = (1 - r_N)\left|\sum_{n=1}^{N} c_n\left(1 + r_N + r_N^2 + \ldots + r_N^{n-1}\right)\right|$$
$$\leqslant (1 - r_N)\sum_{n=1}^{N}\left|c_n\right|\cdot n = \frac{1}{N}\sum_{n=1}^{N_0}\left|nc_n\right| + \frac{1}{N}\sum_{n=N_0+1}^{N}\left|nc_n\right| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} < \frac{\varepsilon}{3},$$

provided that N is sufficiently large. For $N > N_0$, we can also estimate

$$\left|\sum_{n=N+1}^{\infty} c_n r_N^n\right| \leqslant \frac{\varepsilon}{3N}\sum_{n=N+1}^{\infty} r_N^n = \frac{\varepsilon}{3N}\cdot\frac{r_N^{N+1}}{1 - r_N} = \frac{\varepsilon}{3N}\cdot\frac{r_N^{N+1}}{1 - 1 + \frac{1}{N}} < \frac{\varepsilon}{3}.$$

Combining all the above assumptions we conclude that for all sufficient large positive integers N, we have

$$\left|\sum_{n=1}^{N} c_n - \sigma\right| \leqslant \left|\sum_{n=1}^{N} c_n - \sum_{n=1}^{\infty} c_n r_N^n\right| + \left|\sum_{n=1}^{\infty} c_n r_N^n - \sigma\right|$$
$$\leqslant \left|\sum_{n=1}^{N} c_n\left(1 - r_N^n\right)\right| + \left|\sum_{n=N+1}^{\infty} c_n r_N^n\right| + \left|\sum_{n=1}^{\infty} c_n r_N^n - \sigma\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

6. Let $P_r(\vartheta)$ be the Poisson kernel in the unit disk $\mathbb{D}$. Let

$$u(r,\vartheta) = \frac{\partial P_r(\vartheta)}{\partial \vartheta}, \quad 0 \leqslant r < 1, \quad \left|\vartheta\right| \leqslant \pi.$$

Prove that u is harmonic in $\mathbb{D}$ and that for all ϑ

$$\lim_{r \longrightarrow 1-} u(r,\vartheta) = 0.$$

However, u is not identically zero. Why is this not in contradiction with the results given in the lectures?

Solution. The harmonicity of u in $\mathbb{D}$ is rather obvious, for the function $P_r(\vartheta)$ (considered as a function defined in $\mathbb{D}$ via the polar coordinates) is a smooth harmonic function in $\mathbb{D}$ and so

$$\triangle u = \triangle \frac{\partial P_r}{\partial \vartheta} = \frac{\partial}{\partial \vartheta} \triangle P_r \equiv 0.$$

One way to see the harmonicity of the Poisson kernel is to observe that it is the real part of the function $z \longmapsto \frac{1+z}{1-z} \colon \mathbb{D} \longrightarrow \mathbb{C}$ which is analytic in $\mathbb{D}$.

It is easy to get an explicit expression for u in $\mathbb{D}$:

$$u(r,\vartheta) = \frac{\partial P_r}{\partial \vartheta} = \frac{\partial}{\partial \vartheta} \frac{1-r^2}{1-2r\cos\vartheta + r^2} = \frac{2r\,(1-r^2)\sin\vartheta}{\left(1-2r\cos\vartheta + r^2\right)^2}.$$

When $\cos\vartheta \neq 1$, we have

$$u(r,\vartheta) = \frac{2r\,(1-r^2)\sin\vartheta}{\left(1-2r\cos\vartheta + r^2\right)^2} \xrightarrow[r \longrightarrow 1-]{} \frac{2 \cdot 0 \cdot \sin\vartheta}{\left(1-2\cos\vartheta + 1\right)^2} = 0,$$

and when $\cos\vartheta = 1$, we have quite simply $\sin\vartheta = 0$ and

$$u(r,\vartheta) = 0 \xrightarrow[r \longrightarrow 1-]{} 0.$$

The reason with the apparent contradiction with the uniqueness result of the lectures is in the observation that we do not have

$$\lim_{r \longrightarrow 1-} u(r,\vartheta) = 0$$

uniformly in ϑ. One way to see this is to consider the value of u along the curve on which $\cos\vartheta = r$ and $\sin\vartheta = \sqrt{1-r^2}$. On this curve we have

$$u(r,\vartheta) = \frac{2r\,(1-r^2)\sqrt{1-r^2}}{\left(1-2r\cdot r + r^2\right)^2} = \frac{2r}{\sqrt{1-r^2}} \xrightarrow[r \longrightarrow 1-]{} \infty.$$