Introduction to Fourier Analysis
Home assignment 2

1. Assume that $\langle a_n \rangle_{n=1}^N$ and $\langle b_n \rangle_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the k^{th} partial sum of the series $\sum b_n$, and let $B_0 = 0$. Prove the *summation by parts* formula

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$

Solution. It is a straightforward approach to start with the left-hand side and expand the multiplicand b_n in terms of the partial sums B_n. This way we get

$$\begin{aligned}
\sum_{n=M}^{N} a_n b_n &= \sum_{n=M}^{N} a_n (B_n - B_{n-1}) = \sum_{n=M}^{N} a_n B_n - \sum_{n=M}^{N} a_n B_{n-1} \\
&= a_N B_N + \sum_{n=M}^{N-1} a_n B_n - a_M B_{M-1} - \sum_{n=M}^{N-1} a_{n+1} B_n \\
&= a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.
\end{aligned}$$

2. Using the previous exercise prove *Dirichlet's test for convergence of a series*: if the partial sums of the series $\sum_{n=1}^{\infty} b_n$ are bounded, and if $\langle a_n \rangle$ is a sequence of real numbers tending monotonically to 0, then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

Solution. Let $\langle B_n \rangle_{n=1}^{\infty}$ be the sequence of the partial sums of the series $\sum_{n=1}^{\infty} b_n$, let $C \in \mathbb{R}_+$ be a constant such that $|B_n| < C$ for every $n \in \mathbb{Z}_+$, and suppose that an arbitrarily small positive real number ε is given. Let M and N be positive integers with $N > M$ and so large that $a_\ell < \frac{\varepsilon}{3C}$ for all integers $\ell \geqslant M$. Then a simple summation by parts gives the estimate

$$\begin{aligned}
\left| \sum_{n=M}^{N} a_n b_n \right| &= \left| a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n \right| \\
&\leqslant |a_N| \cdot C + |a_M| \cdot C + C \sum_{n=M}^{N-1} |a_{n+1} - a_n|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\varepsilon}{3C}\cdot C + \frac{\varepsilon}{3C}\cdot C + C\left|\sum_{n=M}^{N-1}(a_{n+1}-a_n)\right| \\
&= \frac{2\varepsilon}{3} + C\,|a_N - a_M| \leqslant \frac{2\varepsilon}{3} + C\big|a_M\big| < \varepsilon,
\end{aligned}
$$

thereby proving that the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n b_n$ is a fundamental sequence.

3. Let f be the 2π-periodic *saw tooth function* defined for $\big|x\big| < \pi$ by

$$
f(x) = \begin{cases} -\frac{\pi+x}{2}, & -\pi < x < 0, \\ \frac{\pi-x}{2}, & 0 < x < \pi. \end{cases}
$$

Sketch the graph of f and show that

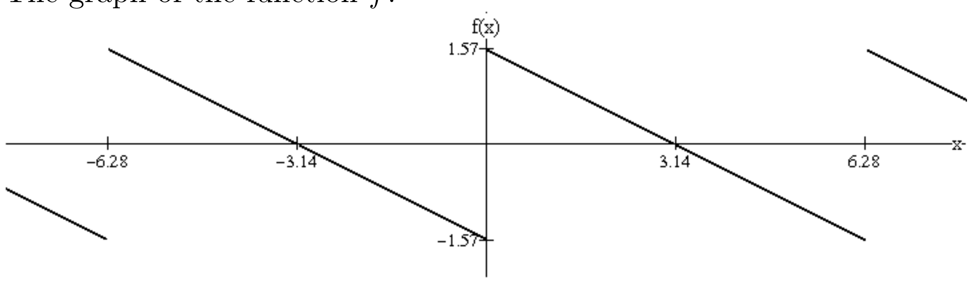
$$
f(x) \sim \frac{1}{2i}\sum_{n\neq 0}\frac{e^{inx}}{n}.
$$

Solution. Since the function f is odd, we have $\widehat{f}(0) = 0$ and, for non-zero integers n,

$$
\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\mathrm{d}x = -\frac{i}{\pi}\int_{0}^{\pi}\left(\frac{\pi-x}{2}\right)\sin nx\mathrm{d}x \\
&= \frac{i}{2\pi}\left(\left.\frac{(\pi-x)\cos nx}{n}\right]_0^{x=\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\mathrm{d}x\right) \\
&= \frac{i}{2\pi}\left(-\frac{\pi}{n} + \frac{1}{n}\cdot\left.\frac{\sin nx}{n}\right]_0^{x=\pi}\right) = -\frac{i}{2n} = \frac{1}{2ni},
\end{aligned}
$$

as required.

The graph of the function f:



4. Show using the Dirichlet test that the Fourier series in the previous exercise converges at every point. What can you say about the sum of the series at the origin in terms of values of f?

Solution. At zero the convergence is obvious, since the symmetric partial sums of the Fourier series of f vanish and therefore the series converges to zero at the origin. This value is the arithmetic mean of the one-sided limits $\pm\frac{\pi}{2}$ which the function f has at the origin.

For fixed $x \in \mathbb{R} \setminus \{0\}$ and all $N \in \mathbb{Z}_+$, we have

$$\left|\sum_{n=1}^{N}\left(e^{inx} - e^{-inx}\right)\right| \leqslant 2\left|\sum_{n=1}^{N} e^{inx}\right| = 2\left|\frac{e^{iNx}-1}{e^{ix}-1}\right| \leqslant \frac{4}{\left|e^{ix}-1\right|},$$

so that the partial sums of the series $\sum_{n=1}^{\infty}\left(e^{inx} - e^{-inx}\right)$ are bounded. Since $\frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$, the Dirichlet test implies that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(e^{inx} - e^{-inx}\right)$$

converges, and the partial sums of this series are precisely the symmetric partial sums of the Fourier series of f at x.

5. Prove that if the series $\sum c_n$ of complex numbers converges and the sum is s, then $\sum c_n$ is Cesàro-summable to s.

Solution. Let $\langle s_n \rangle_{n=1}^{\infty}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} c_n$, and let $\sigma_N = \frac{1}{N}\sum_{n=1}^{N} s_n$ for each $N \in \mathbb{Z}_+$. We know that $s_n \xrightarrow[n \longrightarrow \infty]{} s$ and we have to prove that $\sigma_n \xrightarrow[n \longrightarrow \infty]{} s$ as well.

Let us first suppose that $s = 0$, and suppose that ε is some given arbitrarily small positive real number. Let $N_0 \in \mathbb{Z}_+$ be so large that $\left|s_n\right| < \frac{\varepsilon}{2}$ for all integers $n > N_0$, and let the number N be an integer greater than N_0 and so large that $\frac{1}{N}\left|\sum_{n=1}^{N_0} s_n\right| < \frac{\varepsilon}{2}$. Then

$$\left|\sigma_N\right| = \left|\frac{\sum_{n=1}^{N} s_n}{N}\right| \leqslant \left|\frac{\sum_{n=1}^{N_0} s_n}{N}\right| + \frac{\sum_{n=N_0+1}^{N}\left|s_n\right|}{N} \leqslant \frac{\varepsilon}{2} + \frac{N-N_0}{N}\cdot\frac{\varepsilon}{2} < \varepsilon,$$

and thus $\sigma_N \xrightarrow[N \longrightarrow \infty]{} 0$.

Let us next tackle the general case $s \in \mathbb{C}$. Consider the series

$$-s + c_1 + c_2 + c_3 + \dots.$$

These series converges to zero, and so it is Cesàro-summable to zero as well. This means that

$$\frac{-s + \sum_{n=1}^{N}\left(-s + s_n\right)}{N+1} \xrightarrow[N \longrightarrow \infty]{} 0,$$

so that

$$\frac{N}{N+1} \cdot \frac{\sum_{n=1}^{N} s_n}{N} \xrightarrow[N \longrightarrow \infty]{} s.$$

Let ε be an arbitrarily small positive real number, and let the integer N be so large that

$$\left|\frac{\sigma_N}{N+1}\right| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \left|\frac{N\sigma_N}{N+1} - s\right| < \frac{\varepsilon}{2}$$

The first condition can be assumed since the limit $\frac{N}{N+1}\sigma_N \longrightarrow s$ readily implies that the sequence $\langle \sigma_N \rangle_{N=1}^{\infty}$ is bounded. Now

$$\left|\sigma_N - s\right| \leqslant \left|\sigma_N - \frac{N\sigma_N}{N+1}\right| + \left|\frac{N\sigma_N}{N+1} - s\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so that $\sigma_N \xrightarrow[N \longrightarrow \infty]{} s$.

6. Let

$$L_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|D_N(x)\right| \mathrm{d}x,$$

where D_N is the Dirichlet kernel. Show that for all positive N,

$$L_N \geqslant c \log N,$$

for some positive constant c. Hence $\langle D_N \rangle$ is not a family of good kernels.

Solution. The Dirichlet kernel D_N satisfies the equality

$$D_N(x) = \frac{\sin\left(N + \frac{1}{2}\right)x}{\sin\frac{x}{2}}$$

for all $x \in \mathbb{R} \setminus \{0\}$ and each $N \in \mathbb{Z}_+$. Therefore

$$L_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|D_N(x)\right| \mathrm{d}x = \frac{1}{\pi} \int_{0}^{\pi} \left|\frac{\sin\left(N + \frac{1}{2}\right)x}{\sin\frac{x}{2}}\right| \mathrm{d}x$$

$$\gg \int_{0}^{\pi} \frac{\left|\sin\left(N + \frac{1}{2}\right)x\right| \mathrm{d}x}{x} = \int_{0}^{\pi\left(N+\frac{1}{2}\right)} \frac{\left|\sin y\right| \mathrm{d}y}{y} \geqslant \int_{0}^{\pi N} \frac{\left|\sin y\right| \mathrm{d}y}{y}$$

$$\geqslant \sum_{n=1}^{N} \frac{1}{\pi n} \int_{\pi(n-1)}^{\pi n} \left|\sin y\right| \mathrm{d}y \gg \sum_{n=1}^{N} \frac{1}{n} \gg \log N,$$

where "$\gg$" means "$\geqslant c$" for some fixed positive real constant c.