Introduction to Fourier Analysis
Home assignment 1

1. Let f be a 2π-periodic Riemann-integrable function. Show that

$$f(\vartheta) \sim \widehat{f}(0) + \sum_{n\geqslant 1} \Big(\big(\widehat{f}(n) + \widehat{f}(-n)\big)\cos n\vartheta + i\big(\widehat{f}(n) - \widehat{f}(-n)\big)\sin n\vartheta \Big)\,.$$

Solution. All that is required is to write the partial sums of the Fourier series of f, to write the exponential multiplicands in terms of sines and cosines and to rearrange the terms: For each $N \in \mathbb{Z}_+$ and all $x \in \mathbb{R}$, we have

$$\begin{aligned}\sum_{n=-N}^{N} \widehat{f}(n)\, e^{inx} \\ &= \widehat{f}(0) + \sum_{n=1}^{N} \Big(\widehat{f}(n)\big(\cos nx + i\sin nx\big) + \widehat{f}(-n)\big(\cos nx - i\sin nx\big) \Big) \\ &= \widehat{f}(0) + \sum_{n=1}^{N} \Big(\big(\widehat{f}(n) + \widehat{f}(-n)\big)\cos nx \,+\, i\big(\widehat{f}(n) - \widehat{f}(-n)\big)\sin nx \Big).\end{aligned}$$

2. Assume that f is as above and that it is an even function, i.e. $f(-\vartheta) = f(\vartheta)$. Show that $\widehat{f}(n) = \widehat{f}(-n)$, and the Fourier-series of f is the cosine-series.

Solution. This boils down to a simple change of variables:

$$\begin{aligned}\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\mathrm{d}x &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-y)\, e^{iny}\mathrm{d}y \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, e^{iny}\mathrm{d}y = \widehat{f}(-n).\end{aligned}$$

The fact that the Fourier series of f is the cosine-series now follows immediately from the previous exercise.

3. Assume that f is as above and that it is an odd function, i.e. $f(-\vartheta) = -f(\vartheta)$. Show that $\widehat{f}(n) = -\widehat{f}(-n)$, and the Fourier-series of f is the sine-series.

Solution. This problem is analogous to the previous one. This time

$$\begin{aligned}\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\mathrm{d}x &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-y)\, e^{iny}\mathrm{d}y \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, e^{iny}\mathrm{d}y = -\widehat{f}(-n).\end{aligned}$$

4. Define $f\colon [0,\pi] \longrightarrow \mathbb{C}$ by letting $f(\vartheta) = \vartheta\,(\pi - \vartheta)$, and extend it to all $\vartheta \in \mathbb{R}$ as an odd 2π-periodic function. Show that

$$f(\vartheta) = \frac{8}{\pi} \sum_{\substack{k \geqslant 1 \\ k \text{ odd}}} \frac{\sin k\vartheta}{k^3}.$$

Solution. Since f is odd, we have $\widehat{f}(0) = 0$. For non-zero integers n the corresponding Fourier coefficients are obtained via multiple integrations by parts:

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\vartheta)\, e^{-in\vartheta} \mathrm{d}\vartheta = -\frac{i}{\pi} \int_{0}^{\pi} \vartheta\,(\pi - \vartheta) \sin n\vartheta \, \mathrm{d}\vartheta \\
&= \frac{i}{\pi} \cdot \frac{\vartheta\,(\pi - \vartheta)\cos n\vartheta}{n} \Bigg]_0^{\pi} - \frac{i}{\pi n} \int_0^{\pi} (\pi - 2\vartheta) \cos n\vartheta \, \mathrm{d}\vartheta \\
&= 0 - \frac{i}{\pi n} \cdot \frac{(\pi - 2\vartheta) \sin n\vartheta}{n} \Bigg]_0^{\vartheta=\pi} - \frac{2i}{\pi n^2} \int_0^{\pi} \sin n\vartheta \, \mathrm{d}\vartheta \\
&= 0 + 0 + \frac{2i}{\pi n^2} \cdot \frac{\cos n\vartheta}{n} \Bigg]_0^{\vartheta=\pi} = 2i \cdot \frac{(-1)^n - 1}{\pi n^3} = \begin{cases} 0 & \Longleftarrow 2 \mid n, \\ -\frac{4i}{\pi n^3} & \Longleftarrow 2 \nmid n. \end{cases}
\end{aligned}$$

Since the series of the Fourier coefficients converges absolutely and the function f is continuous, the Fourier series of f converges pointwise to f. The claim now follows once the values of the Fourier coefficients are plugged in the formula given by the exercise 1.

5. Let $f(\vartheta) = |\vartheta|$, $\vartheta \in [-\pi, \pi]$. Prove that $\widehat{f}(0) = \frac{\pi}{2}$ and

$$\widehat{f}(n) = \frac{-1 + (-1)^n}{\pi n^2}, \qquad n \neq 0.$$

Solution. Clearly

$$\widehat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\vartheta| \mathrm{d}\vartheta = \frac{1}{\pi} \int_0^{\pi} \vartheta \mathrm{d}\vartheta = \frac{\pi}{2},$$

and for non-zero integers n, we have

$$\begin{aligned}\widehat{f}(n) &= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left|\vartheta\right|e^{-in\vartheta}\mathrm{d}\vartheta = \frac{1}{\pi}\int\limits_{0}^{\pi}\vartheta\cos n\vartheta\,\mathrm{d}\vartheta \\ &= \frac{1}{\pi}\left(\left[\frac{\vartheta\sin n\vartheta}{n}\right]_0^{\vartheta=\pi} - \frac{1}{n}\int\limits_0^{\pi}\sin n\vartheta\,\mathrm{d}\vartheta\right) = -\frac{1}{n\pi}\left(-\left[\frac{\cos n\vartheta}{n}\right]_0^{\vartheta=\pi}\right) \\ &= \frac{\cos n\pi - 1}{\pi n^2} = \frac{(-1)^n - 1}{\pi n^2} = \begin{cases} 0 & \Longleftarrow 2 \mid n, \\ -\frac{2}{\pi n^2} & \Longleftarrow 2 \nmid n. \end{cases}\end{aligned}$$

6. Using the previous exercise show that

$$\sum_{\substack{n\geqslant 1 \\ n \text{ odd}}} \frac{1}{n^2} = \frac{\pi^2}{8}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution. The function f of the previous exercise is continuous and its Fourier series are clearly absolutely convergent. Therefore we may evaluate the Fourier series at the origin to get

$$0 = f(0) = \widehat{f}(0) + \sum_{n=1}^{N}\left(\widehat{f}\,(n) + \widehat{f}(-n)\right) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{n=1 \\ 2\nmid n}}^{\infty}\frac{1}{n^2}.$$

Thus

$$\sum_{\substack{n=1 \\ 2\nmid n}}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}.$$

The value of the other series is easily obtained from this one: since

$$\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{4n^2} = \sum_{\substack{n=1 \\ 2\nmid n}}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8},$$

we must have

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{4}{3}\cdot\frac{\pi^2}{8} = \frac{\pi^2}{6}.$$