Introduction to Fourier–analysis

Home assignment 9

1. Let
$$u(\phi) = \int_{-1}^{1} \phi(x,x)\,dx, \quad \phi \in \mathcal{S}(\mathbb{R}^2).$$
Show that $u \in \mathcal{S}'(\mathbb{R}^2)$ and compute its first order derivatives. What is $(\partial_1 + \partial_2)u$?

2. Prove that
$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2(\pi a)},$$
when a is real and not equal to an integer. **Hint:** Apply Poisson–summation formula to
$$g(x) = \begin{cases} 1 - |x|, \ |x| \le 1 \\ 0, \text{ otherwise.} \end{cases}$$

3. This exercise collects the basic properties of the gamma function.

   (a) The *gamma function* $\Gamma(s)$ is defined by
   $$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}, \quad s > 0.$$
   Prove that this is well defined , i.e. the integral exists as an improper integral.

   (b) Prove that $\Gamma(s+1) = s\Gamma(s)$, $s > 0$. Conclude that for every positive integer n we have $\Gamma(n+1) = n!$.

   (c) Show that
   $$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

4. Define the *zeta–function* by
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \ s > 1.$$

Prove that

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty t^{s/2-1}(\vartheta(t)-1)\,dt,$$

where the *theta–function* is defined by

$$\vartheta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}.$$

5. Let's return to the X-ray transform. For each $(t,\theta) \in \mathbb{R} \times [-\pi,\pi]$ let $L_{t,\theta}$ be the line in the (x,y)–plane defined by

$$x\cos\theta + y\sin\theta = t.$$

Define the X-ray transform for $f \in \mathcal{S}(\mathbb{R}^2)$ by

$$X(f)(t,\theta) = \int_{L_{t,\theta}} f = \int_{-\infty}^{\infty} f(t\cos\theta + u\sin\theta, t\sin\theta - u\cos\theta)\,du.$$

Compute $X(g)$, when

$$g(x,y) = e^{-\pi(x^2+y^2)}.$$

6. Let again X denote the X-ray transform. Show that $f \in \mathcal{S}(\mathbb{R}^2)$ and $X(f) = 0$ implies $f = 0$.