Introduction to Fourier–analysis

Home assignment 8

1. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and $f \in L^1(\mathbb{R}^n)$. Define $g(x) = f(Ax)$, $x \in \mathbb{R}^n$. Prove that if A is invertible, then $g \in L^1(\mathbb{R}^n)$. Also, compute $\hat{g}$.

2. Give an example of a function $f \in L^2$ such that $f \notin L^1$ but $\hat{f} \in L^1$.

3. Let for $0 < r < \infty$,
$$g_r(x) = e^{-r|x|^2},\ x \in \mathbb{R}^n.$$
Compute the convolution $g_{r_1} * g_{r_2}$.

4. Let A be a positive definite symmetric $n \times n$ matrix with real coefficients. Compute the Fourier–transform of $f(x) = e^{-\langle Ax, x\rangle}$, $x \in \mathbb{R}^n$.

5. Let $f \in L^1(\mathbb{R})$, and assume that $\hat{f}$ is continuous and satisfies
$$\hat{f}(\xi) = O(|\xi|^{-1-\alpha})$$
as $|\xi| \to \infty$ for some $0 < \alpha < 1$. Prove that f is Hölder continuous of order α, i.e. there exists a constant M such that
$$|f(x+h) - f(x)| \le M h^{\alpha}$$
for all x, $h \in \mathbb{R}$.

   Hint: Use the Fourier–inversion formula to express $f(x+h) - f(x)$ as an integral involving $\hat{f}$, and estimate this integral separately over the sets $|\xi| \le 1/|h|$ and $|\xi| \ge 1/|h|$.

6. Consider the linear partial differential operator $P(x, \partial)$ given by
$$P(x,\partial)\, u(x) = \sum_{[\alpha| \le m} a_\alpha(x)\, \partial^\alpha u(x),$$
where the C^∞–coefficients a_α and all their derivatives are uniformly bounded on $\mathbb{R}^n$. Prove that $P(x,\partial)$ maps the Schwarz–space $\mathcal{S}(\mathbb{R}^n)$ to itself continuously.