Introduction to Fourier–analysis

Home assignment 7

Note that this time there are only five problems.

1. Let $\phi(x) = |x|, \quad x \in [-1, 1]$. Extend it to $\mathbb{R}$ as a periodic function of period 2. Prove that
$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x)$$
defines a continuous function on $\mathbb{R}$.

2. We continue to study properties of ϕ defined in the first exercise. Fix $x_0 \in \mathbb{R}$. For every positive integer m define a real numeber δ_m by the conditions $|\delta_m| = 4^{-m}/2$ and the sign is fixed by the condition that there are no integers between $4^m x_0$ and $4^m(x_0 + \delta_m)$. Consider the quotient
$$\gamma_n = \frac{\phi(4^n(x_0 + \delta_m)) - \phi(4^n x_0)}{\delta_m}.$$
Prove that
   (a) If $n > m$ then $\gamma_n = 0$.
   (b) If $0 \le n \le m$, then $|\gamma_n| \le 4^n$ and $|\gamma_m| = 4^m$.

3. This continues the study of f and ϕ defined in the previous exercises. Prove that
$$\left|\frac{f(x_0 + \delta_m) - f(x_0)}{\delta_m}\right| \ge \frac{1}{2}(3^m + 1).$$
Conclude that f is not differentiable at x_0.

4. Compute the Fourier–transform of the characteristic function of the interval $[-a, a]$, $a > 0$.

5. Compute the Fourier–transform of the function
$$f(x) = \begin{cases} 1 - |x|, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$