Introduction to Fourier–analysis

Home assignment 6

1. Let $\gamma : [a, b] \to \mathbb{R}^2$ be differentiable parametrization for the closed curve Γ. Prove that it is a parametrization by the arc–length if and only if the length of the curve from $\gamma(a)$ to $\gamma(s$ for all s is equal to $s - a$, i.e.
$$\int_a^s |\gamma'(t)|\, dt = s - a, \quad \text{for all } s \in [a, b].$$
2. Prove that any differentiable curve admits a parametrization by the arc–length.
3. Prove the second part of Weyl's criterion: if a sequence $(\xi)_{i=1}^{\infty}$ is equidistributed then for all $k \in \mathbb{Z} \setminus \{0\}$ we have
$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \xi_n} \to 0 \quad \text{as } N \to \infty.$$
Hint: Imitate the proof in the lectures by first considering an analogous statement for a characteristic function of an interval.
4. Prove that the sequence $(a \ln n)_{n=1}^{\infty}$ is not equidistributed for any $a \in \mathbb{R}$.
5. Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a continuous periodic function with period 1 and that $(\xi_n)_{n=1}^{\infty}$ is an equidistributed sequence in $[0, 1)$. Prove that if in addition
$$\int_0^1 f(x)\, dx = 0,$$
then unifromly in x,
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x + \xi_n) = 0.$$
Hint: Approximate with trigonometric polynomials.
6. Let now $f : \mathbb{R} \to \mathbb{C}$ be a bounded measurable function and assume that
$$\int_0^1 f(x)\, dx = 0.$$
Prove that
$$\lim_{N\to\infty} \int_0^1 |\frac{1}{N} \sum_{n=1}^{\mathbb{N}} f(x + \xi_n)|^2\, dx = 0.$$