Introduction to Fourier–analysis

Home assignment 5

1. Assume that f is an 2π–periodic integrable function. Show that for all $n \in \mathbb{Z}$ we have
$$\hat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\pi/n)e^{-inx}\,dx,$$
and hence
$$\hat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} (f(x) - f(x+\pi/n))e^{-inx}\,dx.$$

2. Assume that the function f above also satisfies *the Hölder–condition with exponent* α,
$$|f(x+h) - f(x)| \le C|h|^\alpha,$$
for some $0 < \alpha \le 1$ and all real x and h. Show that the Fourier–coefficients of f satisfy
$$\hat{f}(n) = O(|n|^{-\alpha}).$$

3. Prove that the result given in the previous exercise is sharp by showing that the function
$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha}e^{i2^k x}$$
where $0 < \alpha < 1$ satisfies the Hölder–condition with exponent α and that
$$\hat{f}(N) = N^{-\alpha},$$
when $N = 2^k$.

4. Assume that f is a 2π periodic function that satisfies a Lipschitz–condition with constant K, i.e.
$$|f(x) - f(y)| \le K|x-y| \quad \text{for all } x,\, y.$$
Define for $h > 0$
$$g_h(x) = f(x+h) - f(x-h).$$

Prove that

$$\frac{1}{2\pi}\int_0^{2\pi} |g(x)|^2\,dx = \sum_{n=-\infty}^{\infty} 4|\sin(nh)|^2|\hat{f}(n)|^2,$$

and show that

$$\sum_{-\infty}^{\infty} 4|\sin(nh)|^2|\hat{f}(n)|^2 \leq K^2h^2.$$

5. Let p be a positive integer and f as in the previous exercise. By choosing $h = \pi/2^{p+1}$ above show that

$$\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \leq \frac{K^2\pi^2}{2^{2p+1}}.$$

6. Let again f be as above. By estimating the sum $\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|$ prove that the Fourier series of f converges absolutely.