Introduction to Fourier–analysis

Home assignment 4

1. Let $f : [-\pi, \pi] \to \mathbb{R}$, $f(\theta) = |\theta|$. Use Parseval's formula to compute the sums
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

2. Let f be the 2π–periodic odd function defined on $[0, \pi]$ by $f(\theta) = \theta(\pi - \theta)$. Prove that
$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960}, \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

3. Show that for α not an integer, the Fourier series of
$$\frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha}, \quad x \in [0, 2\pi],$$
is given by
$$\sum_{n=-\infty}^{\infty} \frac{e^{inx}}{n+\alpha}.$$

4. Assume that the complex sequence (a_n) has the property that
$$\sum a_n b_n < \infty$$
for all complex sequences (b_n) such that $\sum |b_n|^2 < \infty$. Prove that $\sum |a_n|^2 < \infty$.

5. Let X be a Banach space and Y a normed space. Let (Λ_n) be a sequence of bounded linear maps $X \to Y$ which are uniformly bounded, i.e. there exists a constant $M < \infty$ such that for all n we have $\|\lambda_n\| \le M$. Assume that there exists a dense set $E \subset X$ so that $(\Lambda_n x)$ converges in Y for all $x \in E$. Prove, that $(\Lambda_n x)$ converges in Y for all $x \in X$.

6. Let $f \in C(\mathbb{T})$. Prove that
$$\lim_{N\to\infty} S_N f(\theta)/\ln N = 0$$

uniformly. On the other hand, assume that the complex sequence (λ_n), $n \in \mathbb{N}$, satisfies

$$\lim_{n\to\infty} \lambda_n / \ln n = 0.$$

Prove that there exists an $f \in C(\mathbb{T})$ such that the sequence $(S_N f(0)/\lambda_N)$ is unbounded.

Hint: use the previous exercise and a better lower bound for the L^1–norm of the Dirichlet–kernel.