Introduction to Fourier–analysis

Home assignment 3

1. Show that the Fejér–kernel can be written as
$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}.$$

2. Prove that the series $\sum_{k=0}^{\infty}(-1)^k(1+k)$ is not Césaro–summable.

3. Prove that if the series $\sum c_n$ of complex numbers is Césaro–summable, and the sum is s, then $\sum c_n$ is Abel –summable to s.

4. Under certain conditions one can reverse the summability results, i.e. from Abel– or Césaro–summability deduce the summability of the original series. These kind of theorems are known as *Tauberian–theorems.* As an example, assume that the sequence $\{c_n\}$ of complex numbers satisfies $nc_n \to 0$ as $n \to \infty$, and that it is Césaro–summable to σ. Prove that $\sum c_n = \sigma$.

5. Again, assume that the sequence $\{c_n\}$ of complex numbers satisfies $nc_n \to 0$ as $n \to \infty$, but now that it is Abel–summable to σ. Prove that $\sum c_n = \sigma$.

6. Let $P_r(\theta)$ be the Poisson kernel in the unit disk $\mathbb{D}$. Let
$$u(r,\theta) = \frac{\partial P_r(\theta)}{\partial\theta},\ 0 \le r < 1,\ |\theta| \le \pi.$$
Prove that u is harmonic in $\mathbb{D}$ and that for all θ
$$\lim_{r\to 1-} u(r,\theta) = 0.$$
However, u is not identically zero. Why is this not a contradiction with the results given in the lectures?