Introduction to Fourier–analysis

Home assignment 2

1. (Stein - Shakarchi, ex.2.7 (a)) Assume $\{a_n\}_{n=1}^N$ and $\{a_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the k^{th} partial sum of the series $\sum b_n$, and let $B_0 = 0$. Prove the *summation by parts* formula

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$

2. (Stein - Shakarchi, ex.2.7 (b)) Using the previous exercise prove the *Dirichlet's test for convergence* of a series: if the partial sums of the series $\sum_{n=1}^{\infty} b_n$ are bounded, and if $\{a_n\}$ is a sequence of real numbers tending monotonically to 0, then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

3. (Stein - Shakarchi, ex.2.8) Let f be the 2π–periodic *saw tooth function* defined for $|x| < \pi$ by

$$f(x) = \begin{cases} -(\pi + x)/2, \; -\pi < x < 0 \\ (\pi - x)/2, \; 0 < x < \pi \end{cases}$$

   Sketch the graph of f, and show that

$$f(x) \sim \frac{1}{2i} \sum_{n \neq 0} \frac{e^{inx}}{n}.$$

4. (Stein - Shakarchi, ex.2.8 continued) Show using the Dirichlet test that the Fourier series in the previous exercise coverges at every point. What can you say about the sum of the series at the origin in terms of values of f?

5. (Stein - Shakarchi, ex.2.12) Prove that if the series $\sum c_n$ of complex numbers converges, and the sum is s, then $\sum c_n$ is Cesáro –summable to s. **Hint**: Reduce the proof to the case $s_n \to 0$.

6. (Stein - Shakarchi, Problem 2.2) Let

$$L_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(x)| \, dx,$$

where D_N is the Dirichlet kernel. Show that for all positive N,

$$L_N \geq c \ln N,$$

for some positive constant c. Hence $\{D_N\}$ is not a family of good kernels. **Hint**: Change variables and prove that

$$L_N \geq c \int_{\pi}^{N\pi} \frac{|\sin\theta|}{|\theta|}\, d\theta + o(1).$$

Then estimate the integrals over intervals $[k\pi, (k+1)\pi]$ separately and compare to the harmonic series.