Introduction to Fourier–analysis

Home assignment 1

1. Let f be 2π–periodic Riemann–integrable function. Show, that
$$f(\theta) \sim \hat{f}(0) + \sum_{n\geq 1} [\hat{f}(n) + \hat{f}(-n)] \cos(n\theta) + i[\hat{f}(n) - \hat{f}(-n)] \sin(n\theta).$$

2. Assume that f is as above and that it is an even function, i.e $f(-\theta) = f(\theta)$. Show that $\hat{f}(n) = \hat{f}(-n)$, and the Fourier series of f is the cosine–series.

3. Assume that f is as above and that it is an odd function, i.e $f(-\theta) = -f(\theta)$. Show that $\hat{f}(n) = -\hat{f}(-n)$, and the Fourier series of f is the sine–series.

4. Define $f : [0, \pi] \to \mathbb{C}$ by letting $f(\theta) = \theta(\pi - \theta)$, and extend it to all $\theta \in \mathbb{R}$ as an odd 2π–periodic function. Show that
$$f(\theta) = \frac{8}{\pi} \sum_{k \,\mathrm{odd},\, k\geq 1} \frac{\sin(k\theta)}{k^3}.$$

5. Let $f(\theta) = |\theta|$, $\theta \in [0, 2\pi]$. Prove that $\hat{f}(0) = \pi/2$ and
$$\hat{f}(n) = \frac{-1 + (-1)^n}{\pi n^2}, \; n \neq 0.$$

6. Using the previous exercise show that
$$\sum_{n \,\mathrm{odd},\, n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{8}, \; \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$