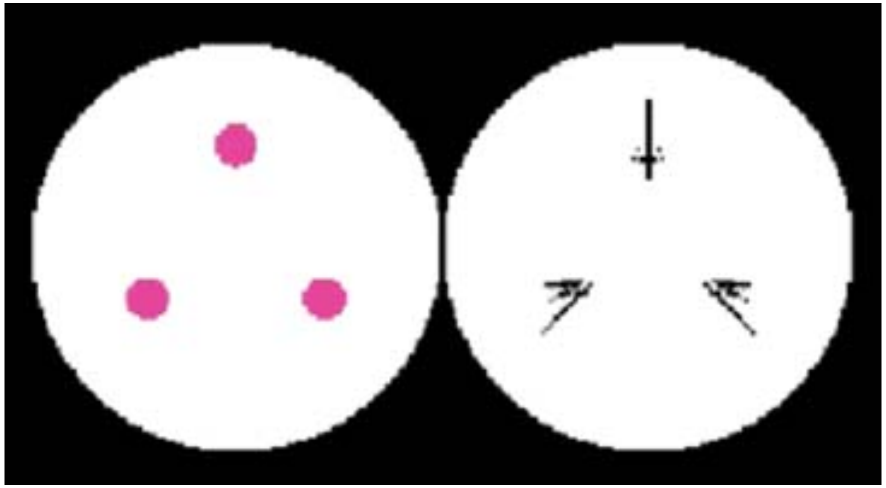
Computational Inverse Problems



Spring 2010
Department of Mathematics and Statistics
University of Helsinki

Lecturers: Matti Lassas (analytic inversion)
Samuli Siltanen (computerized inversion)
Assistant: Lauri Oksanen

Finnish Centre of Excellence in Inverse Problems Research

OULUN YLIOPISTO
UNIVERSITY of OULU

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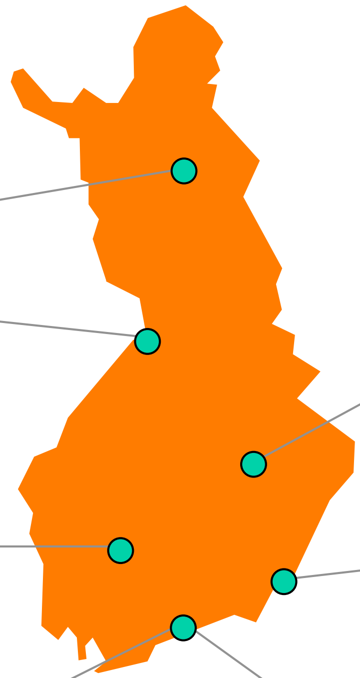

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HELSINKI UNIVERSITY OF TECHNOLOGY

http://math.tkk.fi/inverse-coe/

What are inverse problems?

Inverse problem: Image deblurring

Direct and inverse problem of image deblurring

Direct problem:

Given a sharp photograph, what would the blurred version of the image look like?

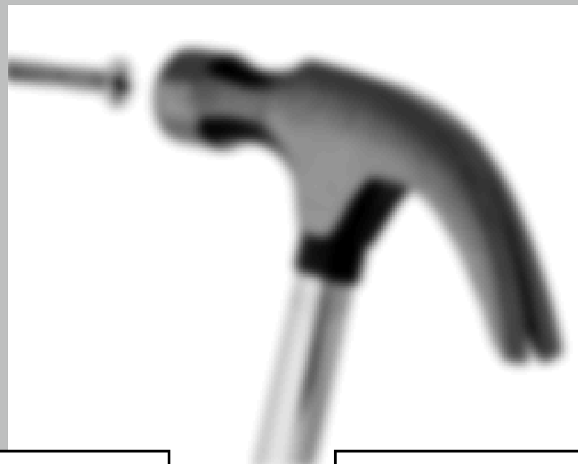
Inverse problem:

Given a blurred photograph, reconstruct the sharp image

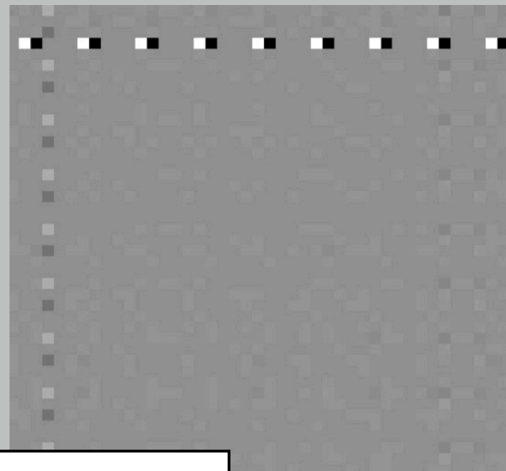
The inverse problem is more difficult

Original image

Blurred image



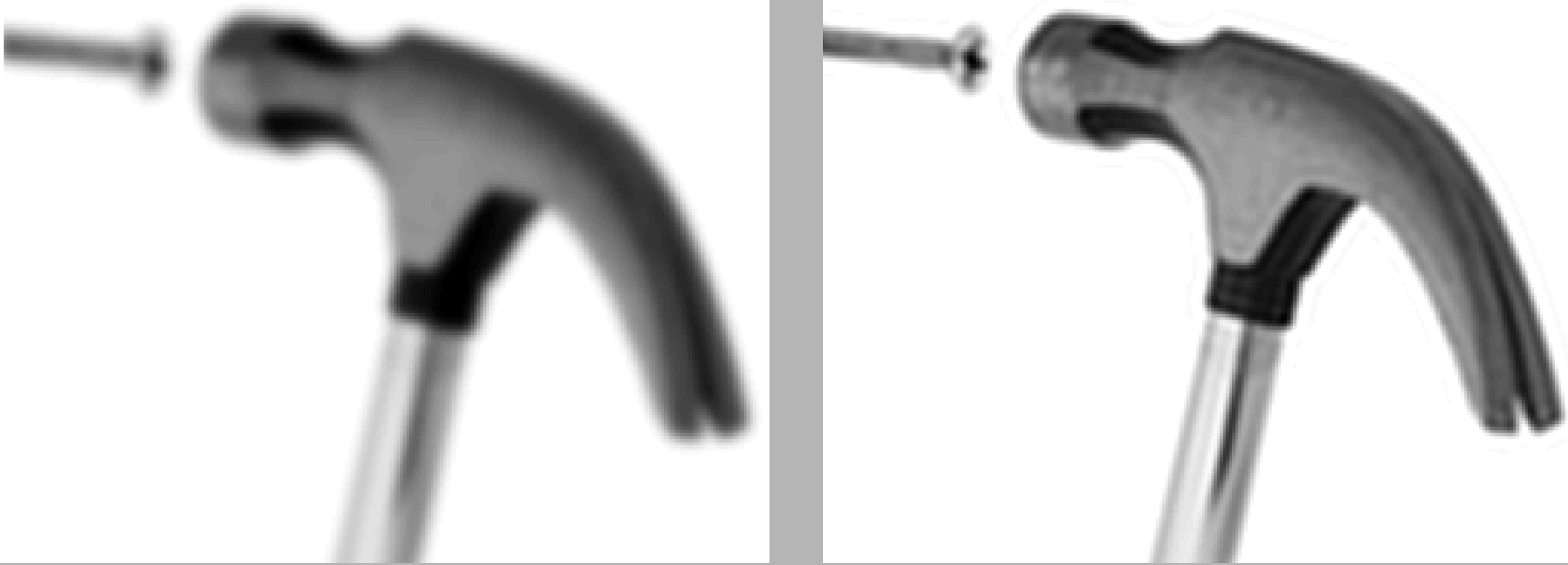
Simple inversion



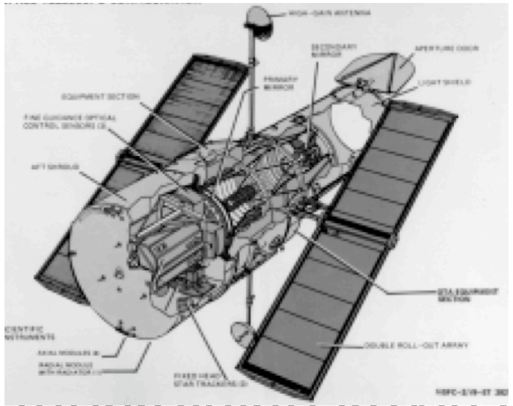
Direct problem

Inverse problem

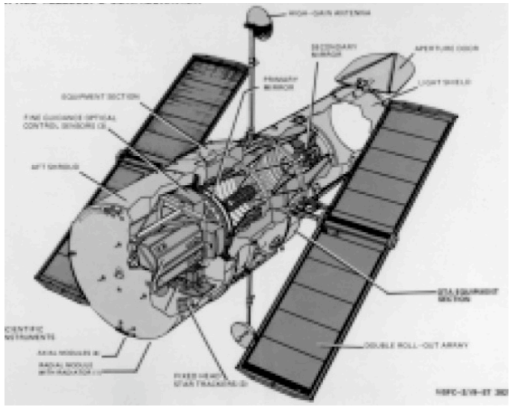
With properly regularized inversion we can sharpen the photograph

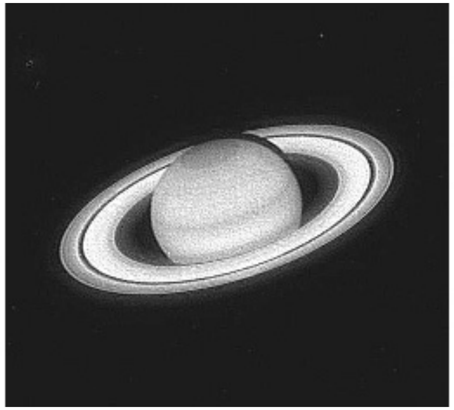


The Hubble space telescope had a flaw in its mirror, resulting in blurred images



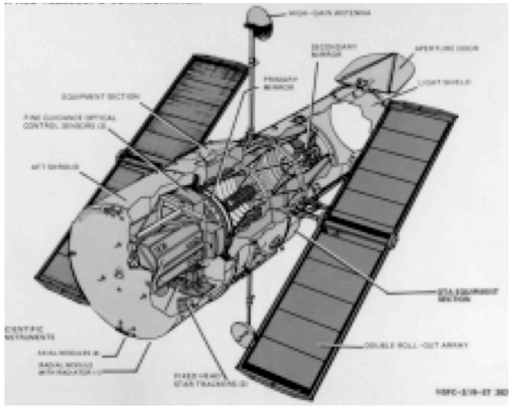
The mirror flaw was compensated by a deconvolution algorithm

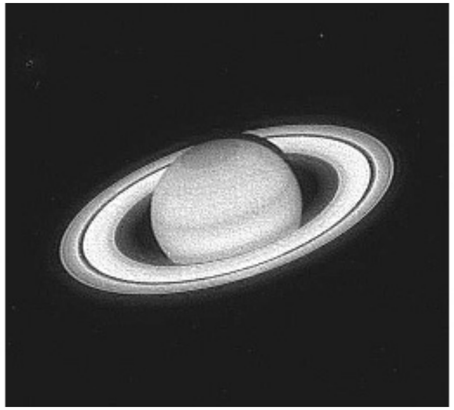




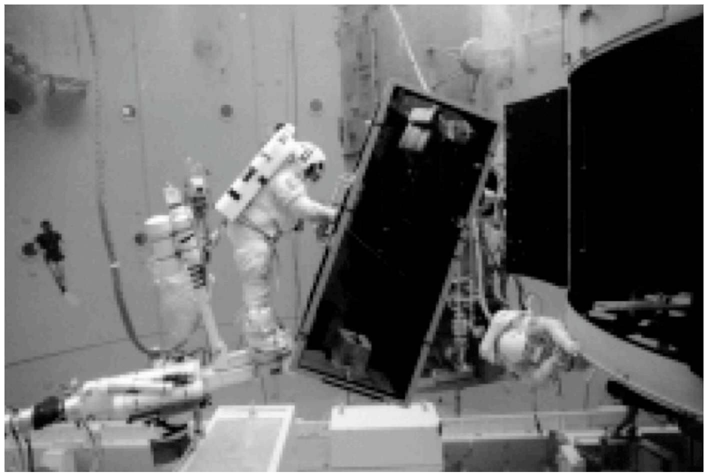
Source: NASA, Quarktet

The mirror flaw was compensated by a deconvolution algorithm





The mirror was replaced in 1993. However, even the new sharp images could be further enhanced with deconvolution!



Source: NASA, Quarktet

Inverse problem: Computerized tomography

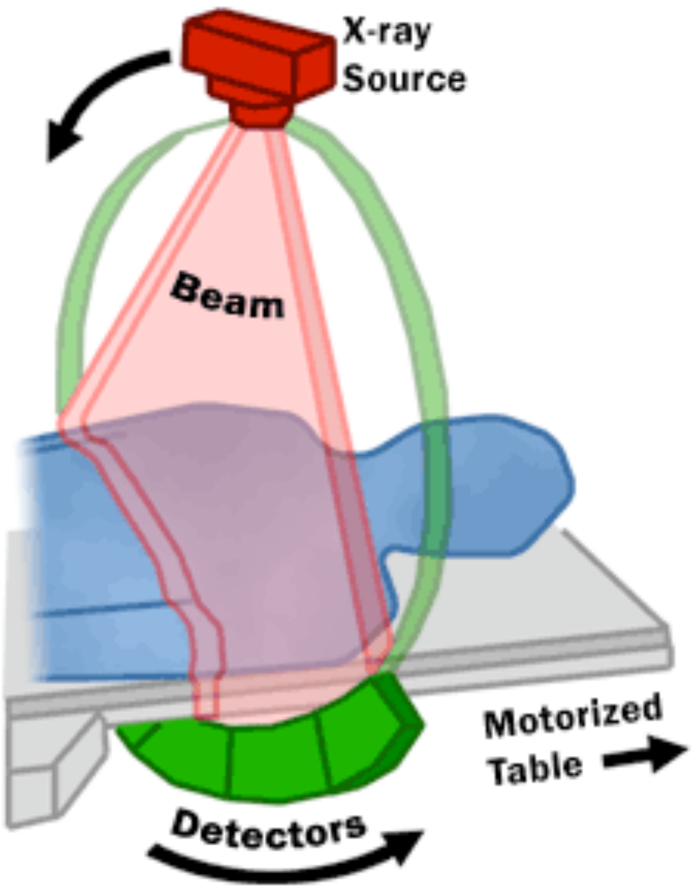
Direct problem:

If the inner structure of a person is known, what would X-ray images of her look like?

Inverse problem:

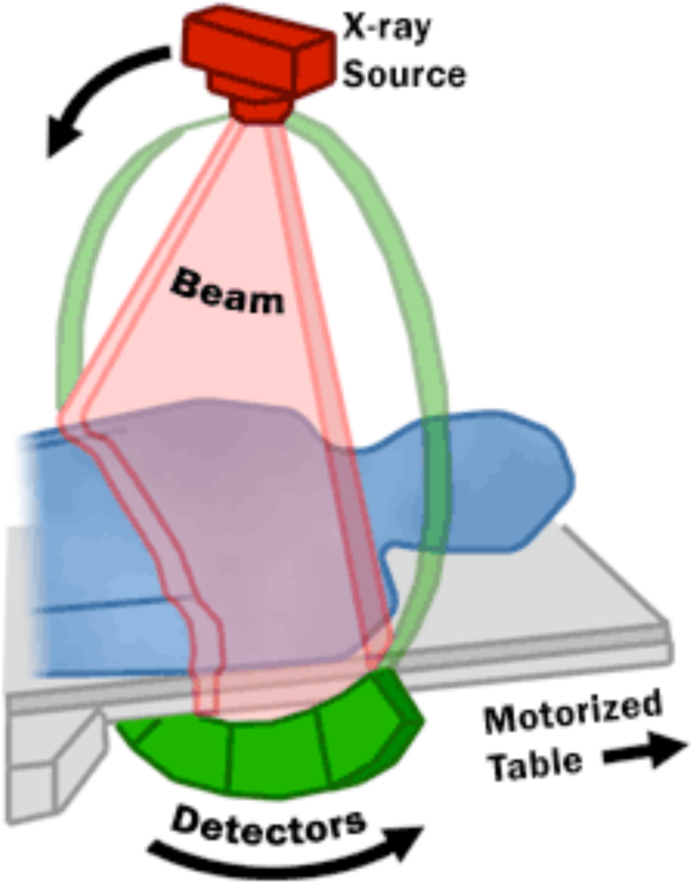
Given X-ray images from all around the body, what is the inner 3-D structure?

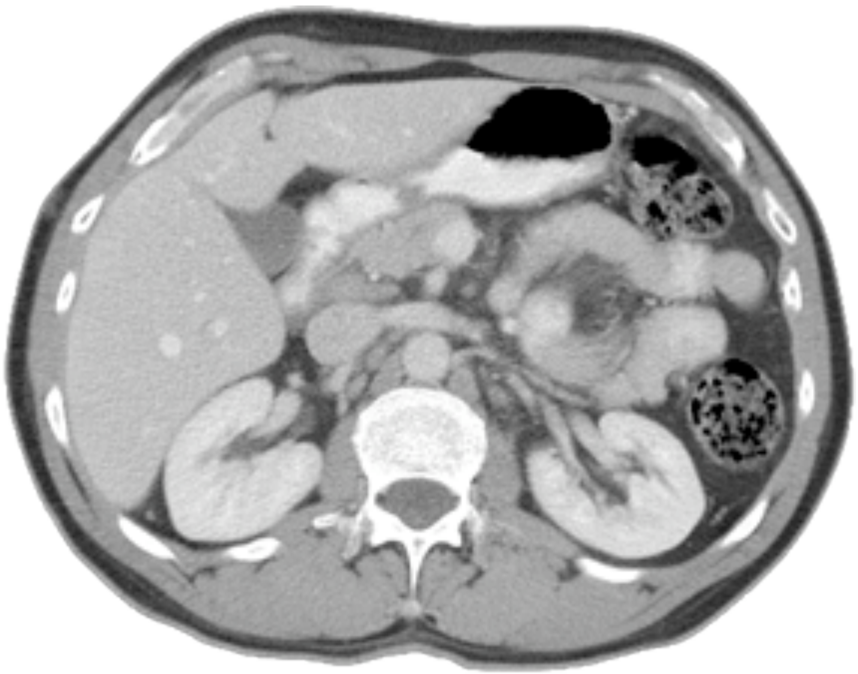
Traditionally, CT data is collected slice by slice



Images from http://www.fda.gov/cdrh/ct/what.html

Using a reconstruction algorithm, inner structure in the slice is revealed





Johann Radon (1887-1956)

$$f(x) = \frac{1}{4\pi^2} \int_{S^1} \int_{\mathcal{R}} \frac{\frac{d}{ds}(Rf)(\theta, s)}{x \cdot \theta - s} ds d\theta$$

Filtered back-projection

$$f(x) = \frac{1}{4\pi^2}\int_{S^1}\int_{\mathcal{R}} \frac{\frac{d}{ds}(Rf)(\theta, s)}{x\cdot\theta - s} ds d\theta$$

Filtered back-projection (FBP) is mathematical technology used on a daily basis in hospitals around the world.

The quality of 3D reconstruction using FBP is excellent.

Nobel prize was awarded to Hounsfield and Cormack 1979.

However, a comprehensive data set is mandatory for FBP.

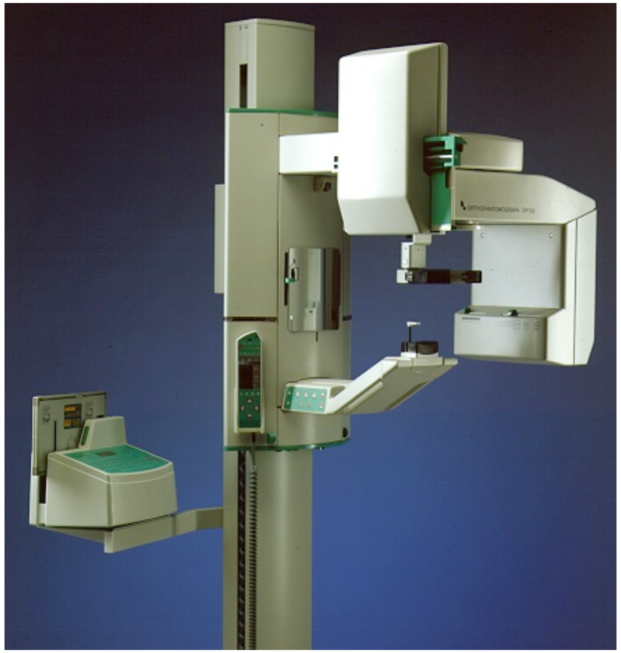
A series of projects started in 2001 aiming for a new type of low-dose 3D imaging

The goal was a mathematical algorithm with

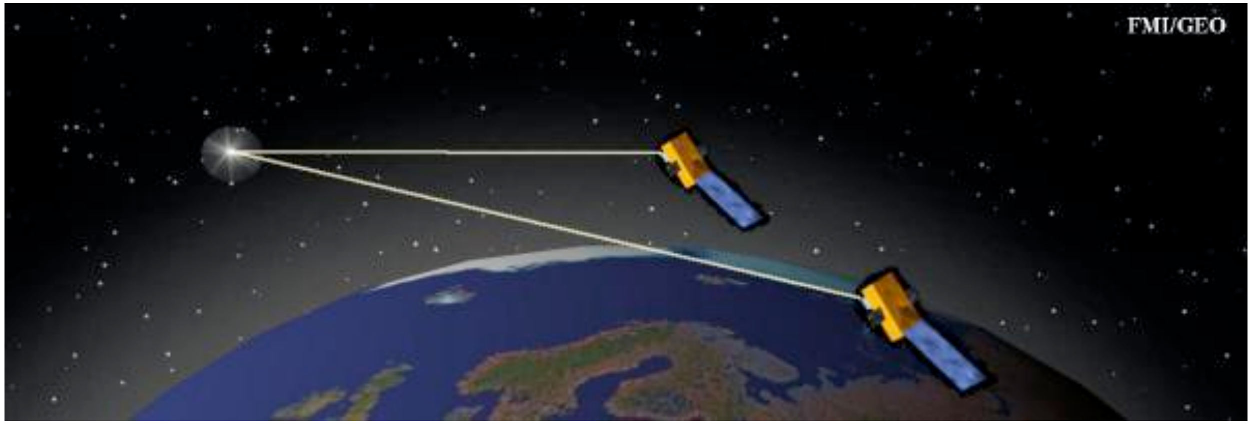
Input: small number of digital X-ray images taken with any X-ray device

Output: three-dimensional reconstruction with quality good enough for the clinical task at hand

Products of Instrumentarium Imaging in 2001:



Inverse problem: ozone layer tomography



Direct problem:

If the ozone profile of the atmosphere were known, what star occultation measurements would we get?

Inverse problem:

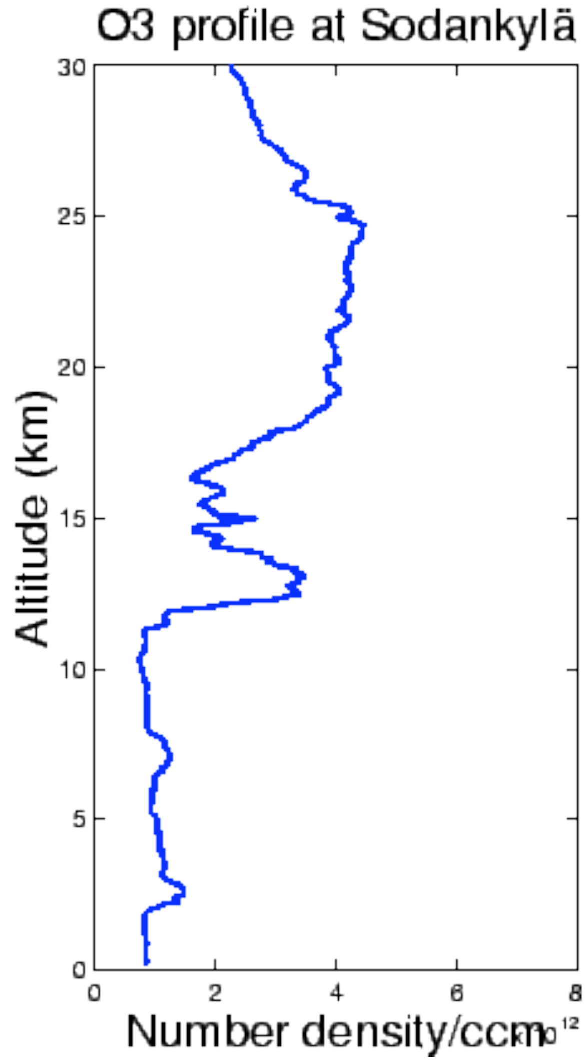
Given star occultation measurements , what is the ozone profile?

Show animation of measurement!

http://envisat.esa.int/instruments/gomos/descr/flash.html

As a result we get ozone density as function of altitude

This inverse problem is mathematically the same than the CT problem, except with limited data



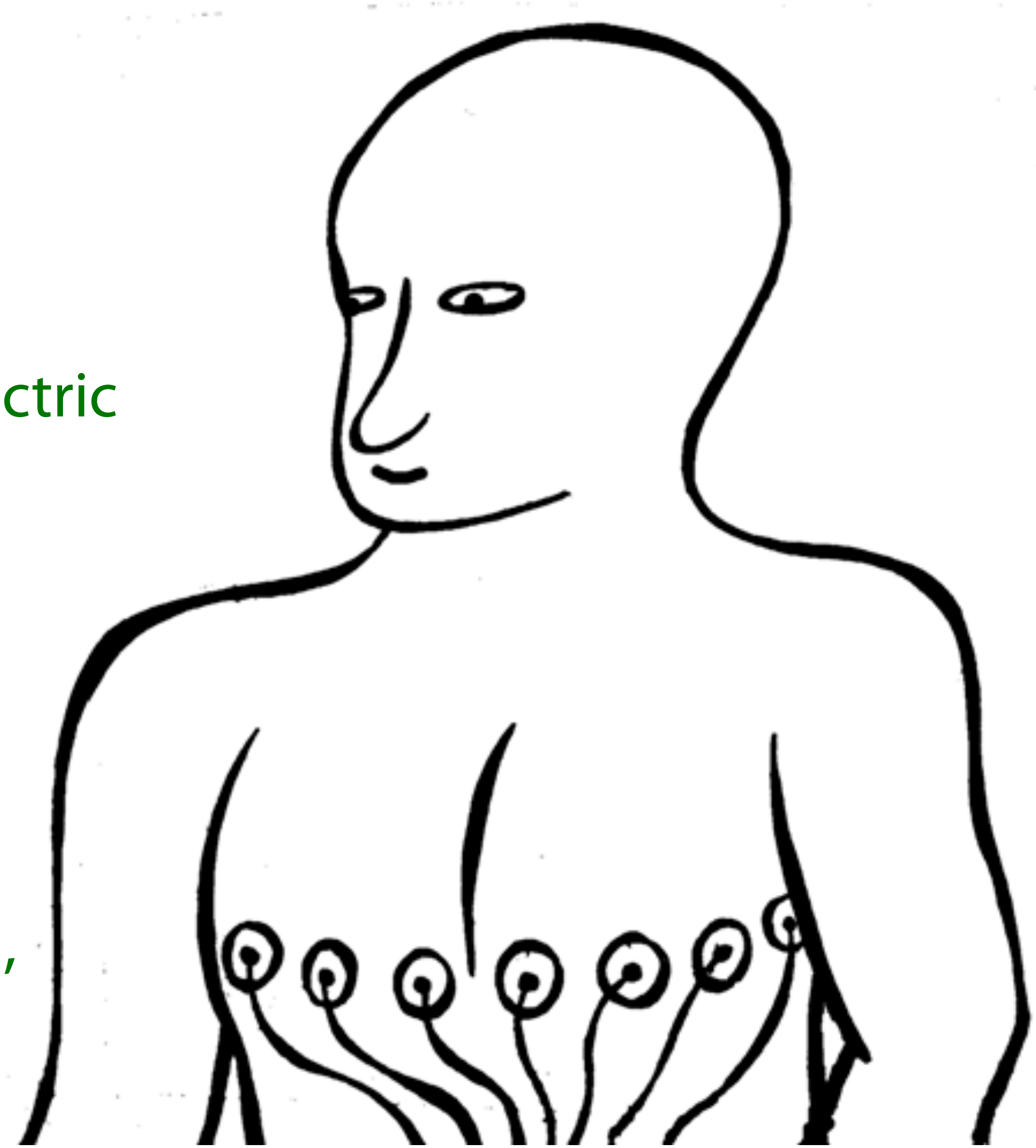
Sources:
European Space Agency
Finnish Meteorological Institute
Envisat and GOMOS projects
http://www.fmi.fi/tutkimus_otsoni/otsoni_26.html
http://envisat.esa.int/handbooks/gomos/CNTR2.htm

Fourth example of inverse problems: Electrical impedance tomography

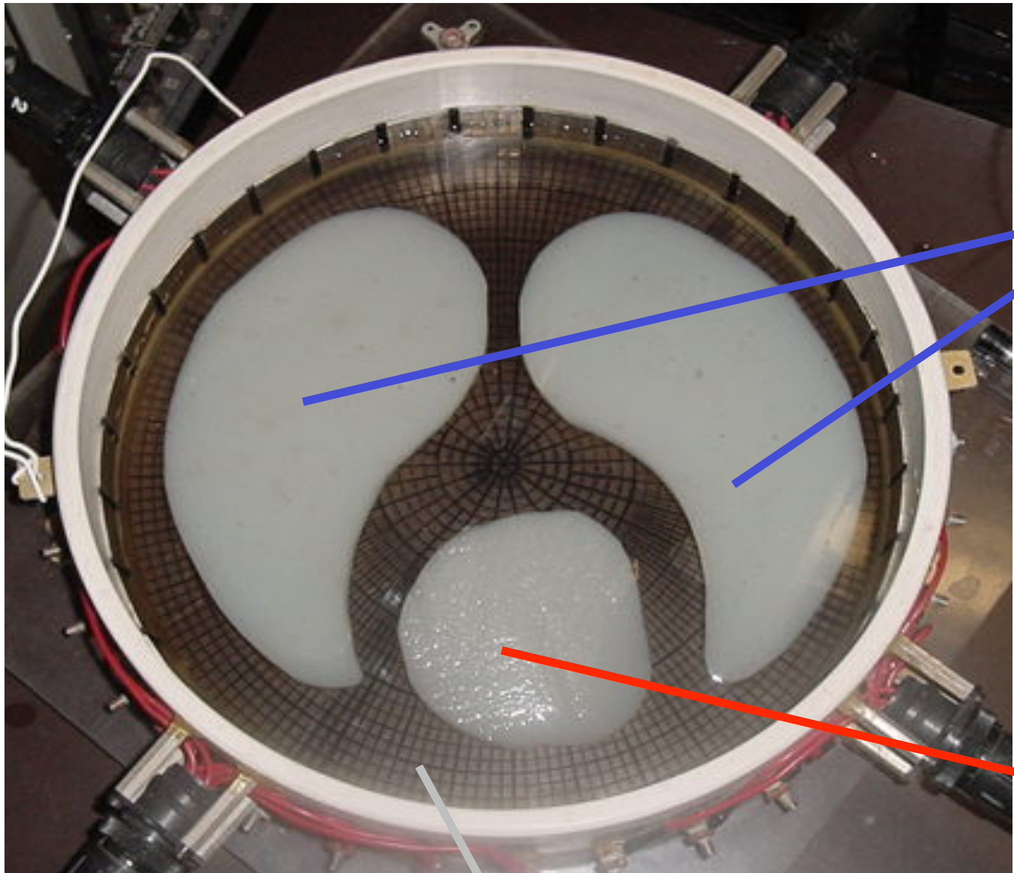
Feed electric currents
through electrodes,
measure voltages

Reconstruct the image of electric
conductivity
in a two-dimensional slice

Applications:
monitoring heart and lungs
of unconscious patients,
detecting pulmonary edema,
enhancing ECG and EEG



At the RPI lab, we construct a chest phantom consisting of saline and agar

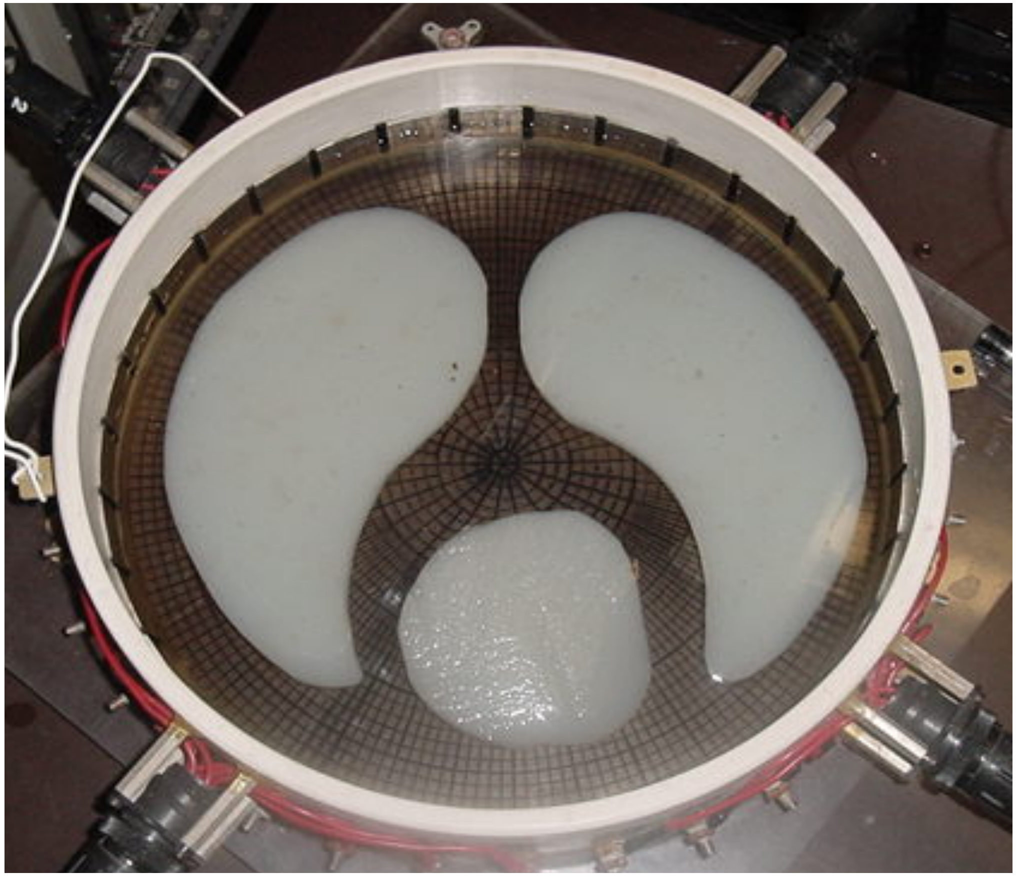


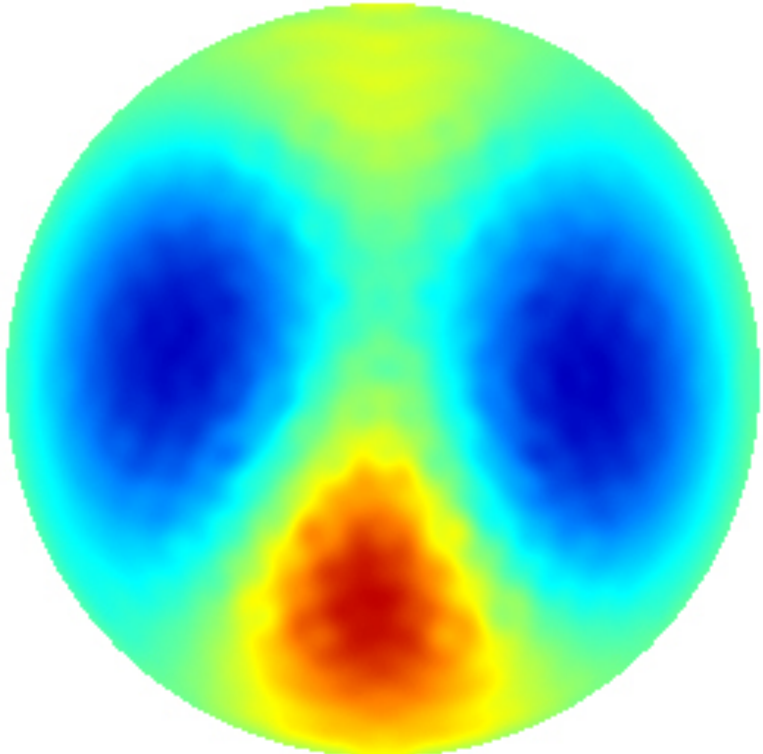
"Lungs" with lower conductivity than background (240 mS/m)

"Heart" with higher conductivity than background (750 mS/m)

Background of salt water, conductivity 424 mS/m.
Diameter of the tank is 30cm.

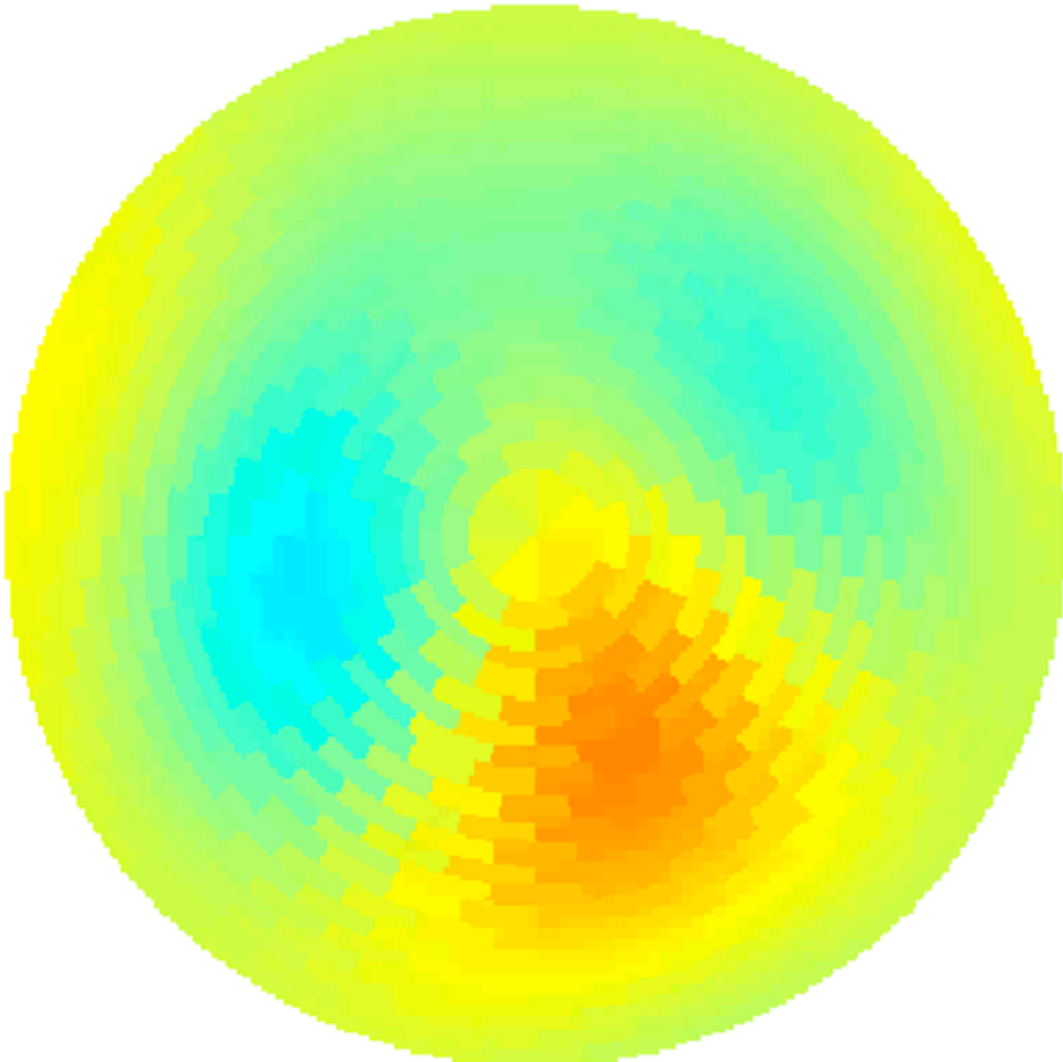
Reconstruction from phantom data



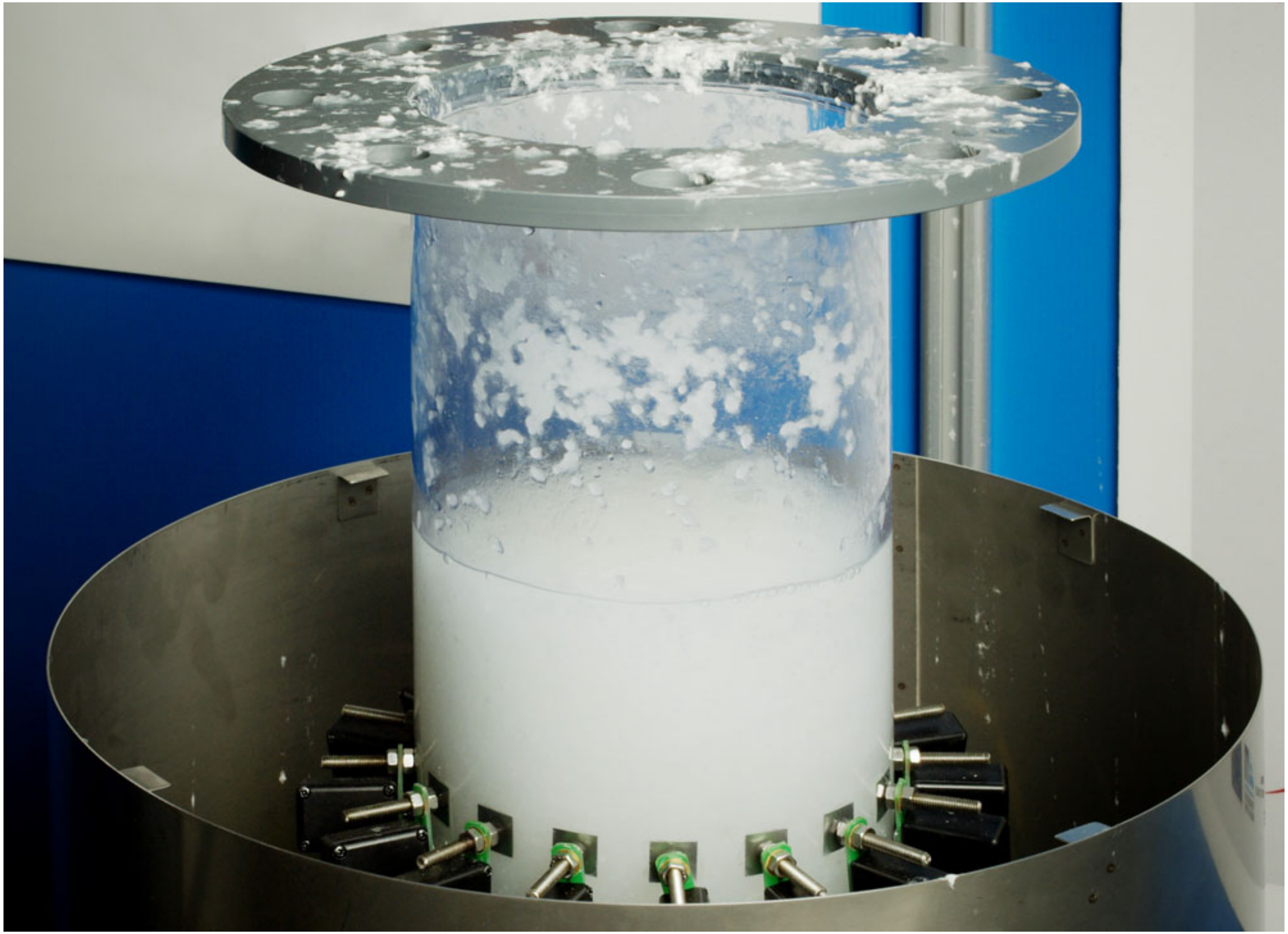


This example is from
Isaacson, Mueller, Newell and Siltanen 2004
IEEE Transactions on Medical Imaging 23, pp. 821-828

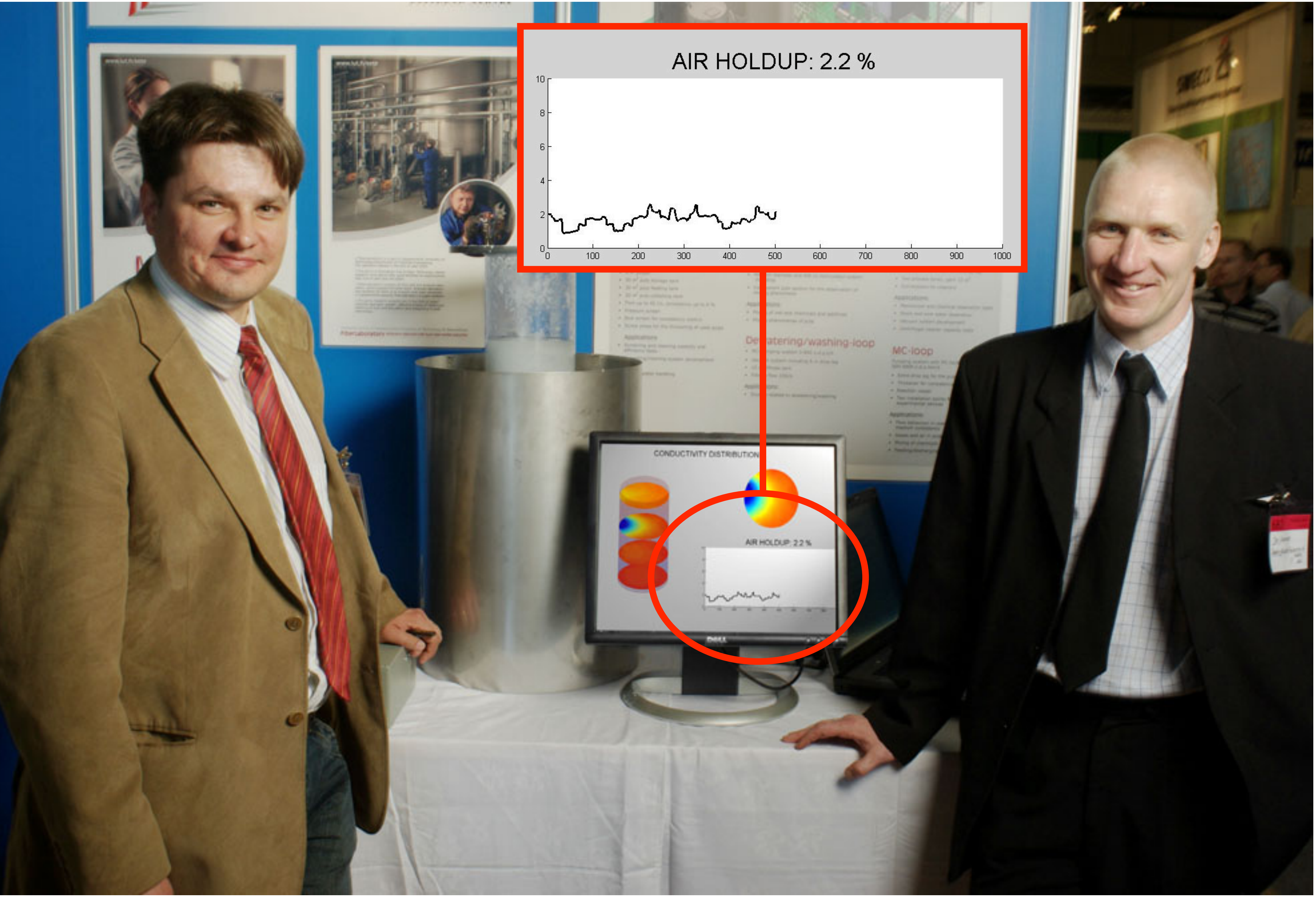
Reconstruction from data collected from a living person



EIT can be used as well in industrial process monitoring

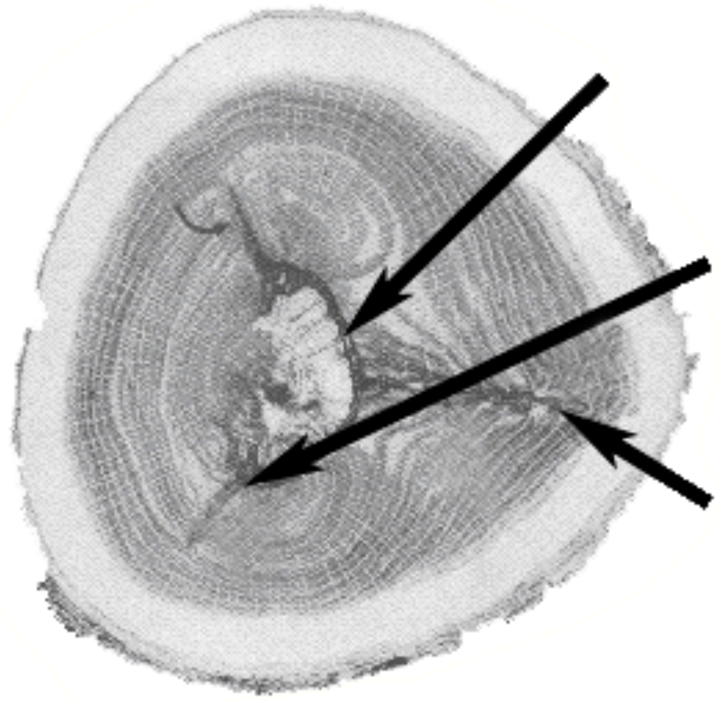


Lasse Heikkinen and Jari Kourunen
University of Kuopio, Finland



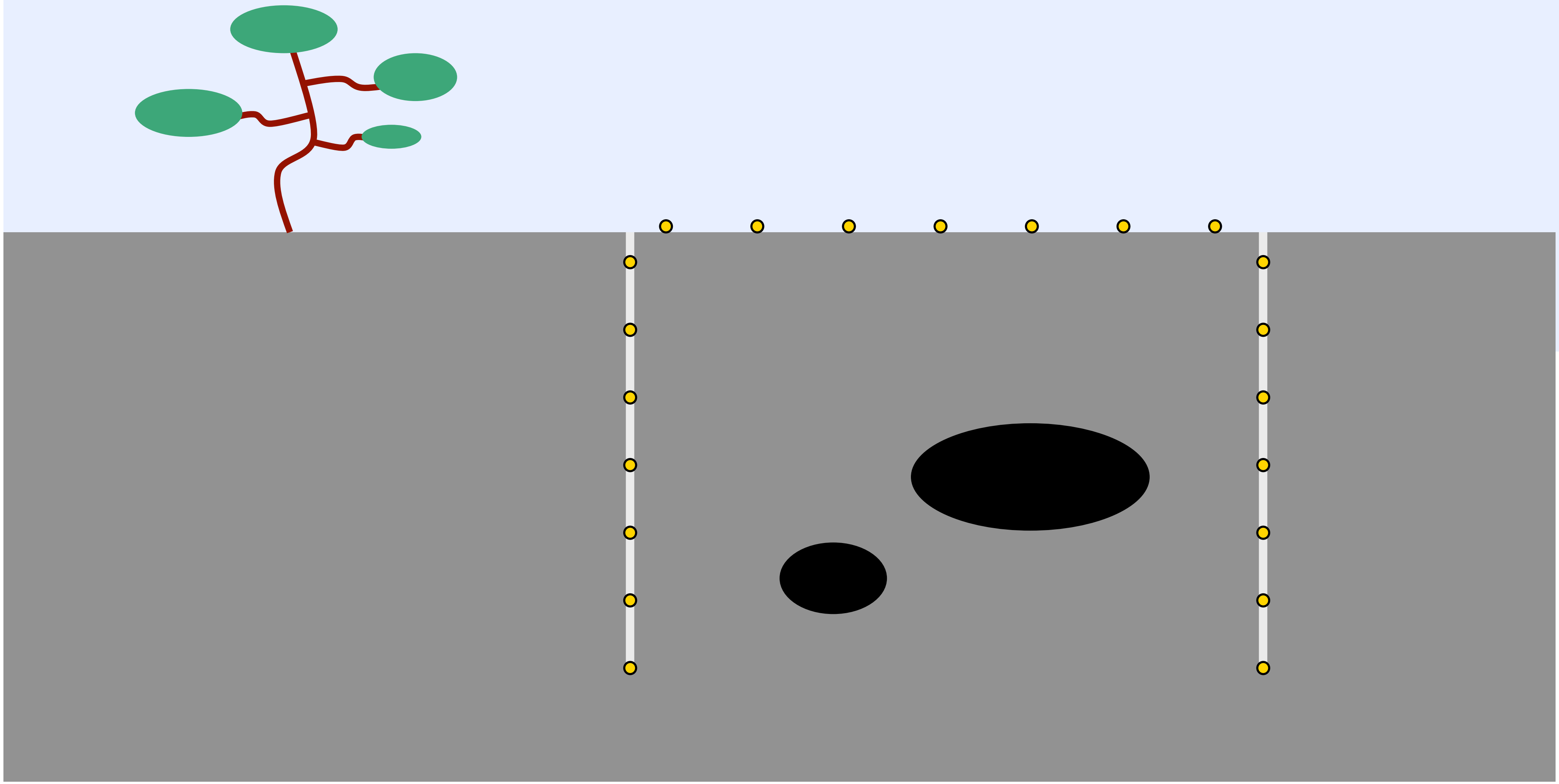
AIR HOLDUP: 2.2 %
10
8
6
4
2
0
0
100
200
300
400
500
600
700
800
900
1000
CONDUCTIVITY DISTRIBUTION
MC-loop

EIT can be used for finding defects in materials



Shigo, A.L., 1983. Tree Defects: A Photo Guide. USDA Forest Service, No. Cent. For. Exp. Sta., GTRNE-82.

Geological sensing of oil or metals is another application of EIT



Inverse problem: Shape optimization

In mechanical engineering, it is important to design parts that are optimal with respect to

1. Mechanical properties: rigidity, resistance
2. Weight
3. Cost

Direct and inverse problem of shape optimization

Direct problem:

Given a mechanical part, find its structural properties, weight and cost

Inverse problem:

Find the shape that globally minimizes cost and weight, still giving optimal performance

Show animation of optimal chair!

http://www.cmap.polytechnique.fr/~optopo/html/chaise_en.html

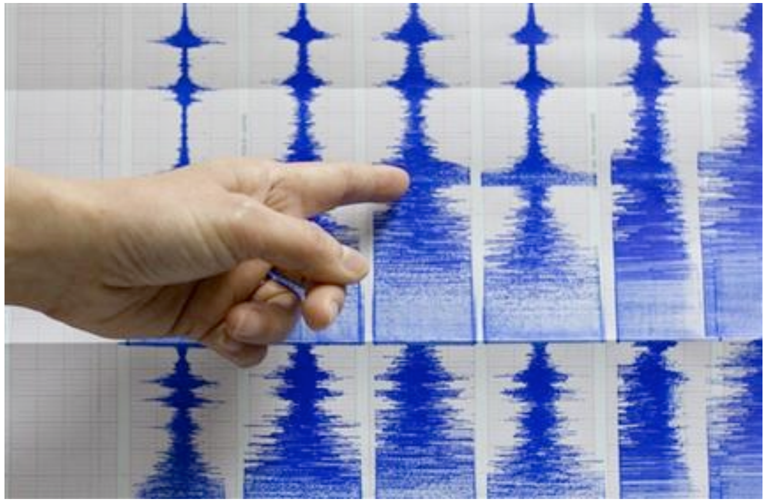
Sixth example of inverse problems: Recovering the inner structure of Earth

Direct problem:

Given the inner structure of Earth, predict vibrations caused by an earthquake

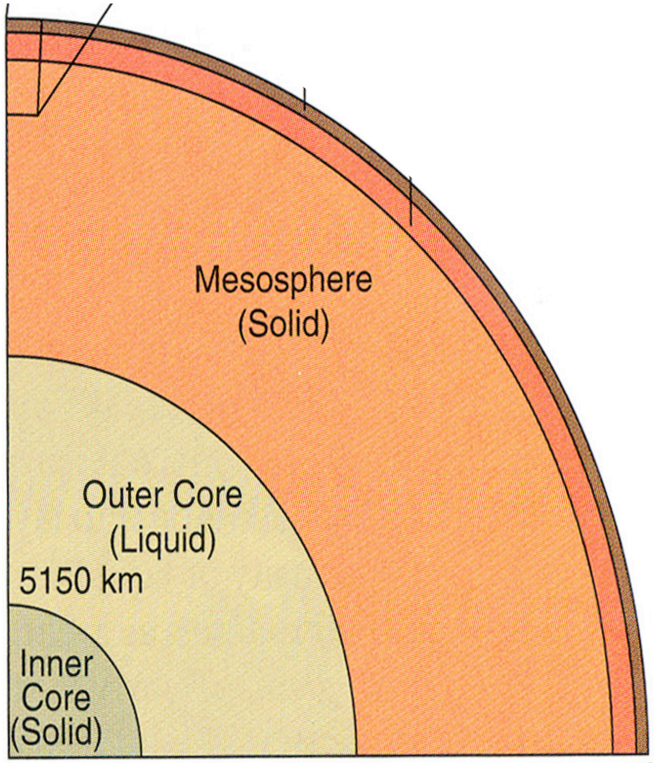
Inverse problem:

Given earthquake data around the world, find sound speed distribution inside Earth



Inverse Problem

Direct problem



(University of New Hampshire)

Inverse problem: pricing financial instruments

The Black-Scholes equation is the mathematical model behind option pricing

Under "perfect market" assumptions
(liquidity, absence of arbitrage and transaction cost)
the call price C satisfies

$$\partial_t C + \frac{1}{2}\sigma^2(t,S)S^2\partial_S^2 C + r(S\partial_S C - C) = 0,$$
$$C(S,T) = (S-K)^+,$$

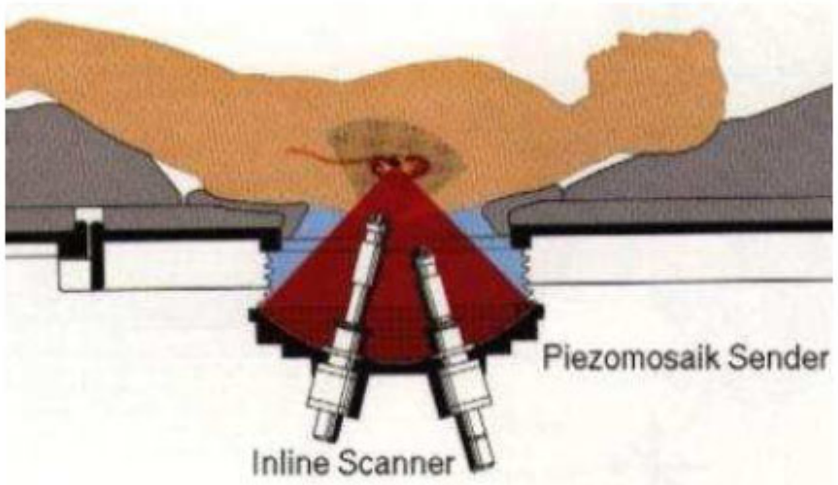
where r is the constant interest rate on a riskless investment.

The inverse problem is to determine the local volatility $\sigma(S,t)$
from a set of noisy call prices:

$$\{C^\delta(S,t;K,T)\}_{(K,T)\in\mathcal{I}}$$

Fifth example of inverse problems: Ultrasound therapy

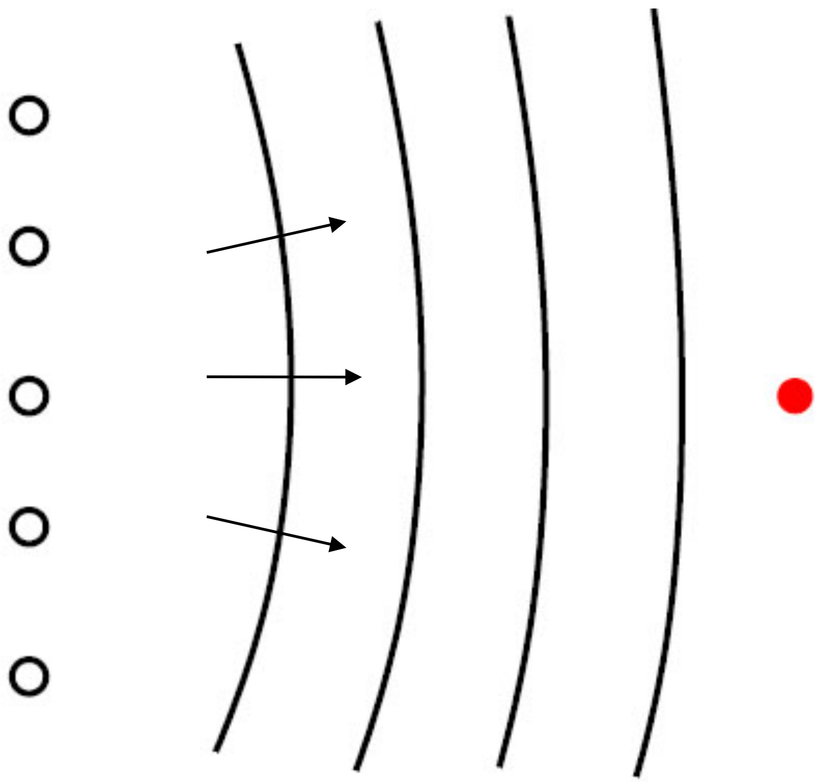
Crushing kidney stones
using ultrasound
(Riedlinger, Karlsruhe)



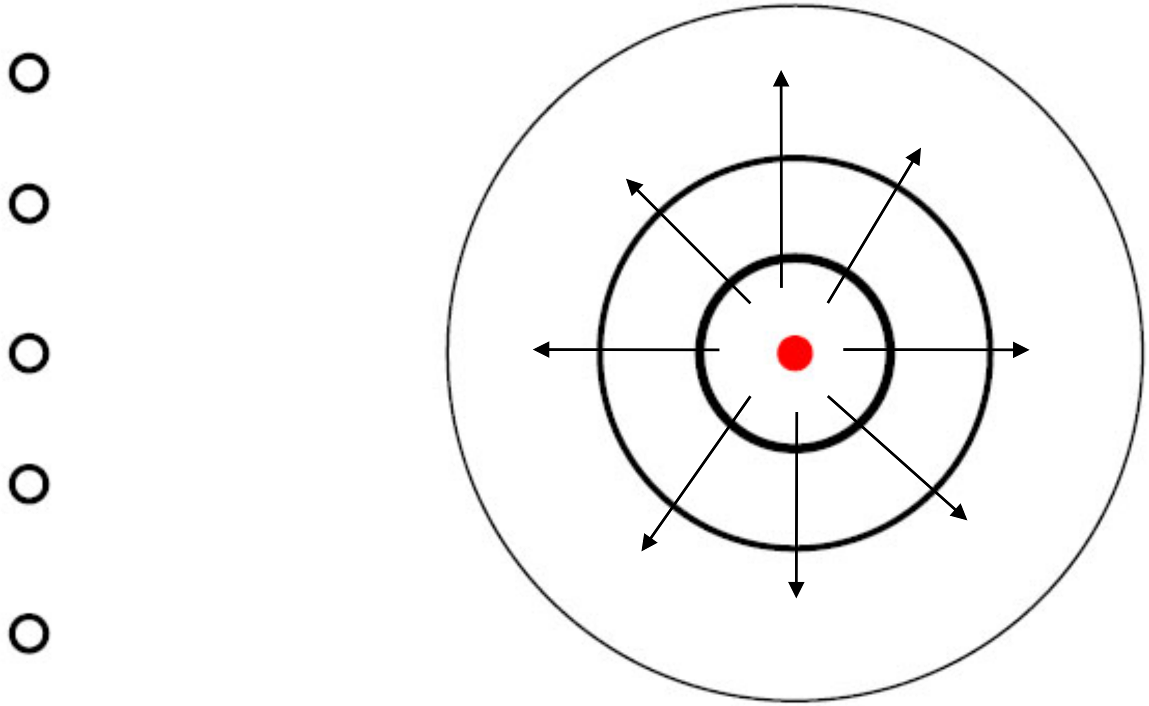
Focusing wave
(Kaipio, Kuopio)

One possibility to focus waves is *time reversal*

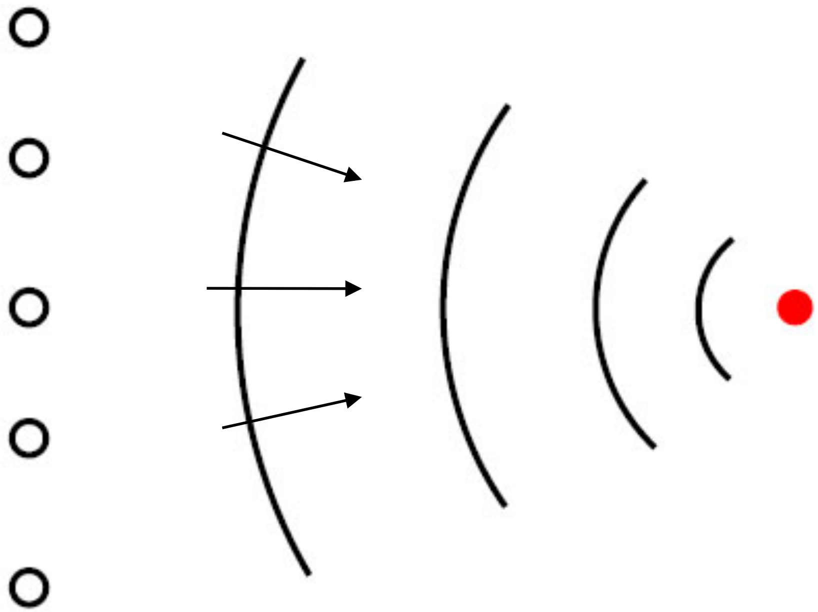
1. Send a probing signal



1. Send a probing signal
2. Measure and record the response

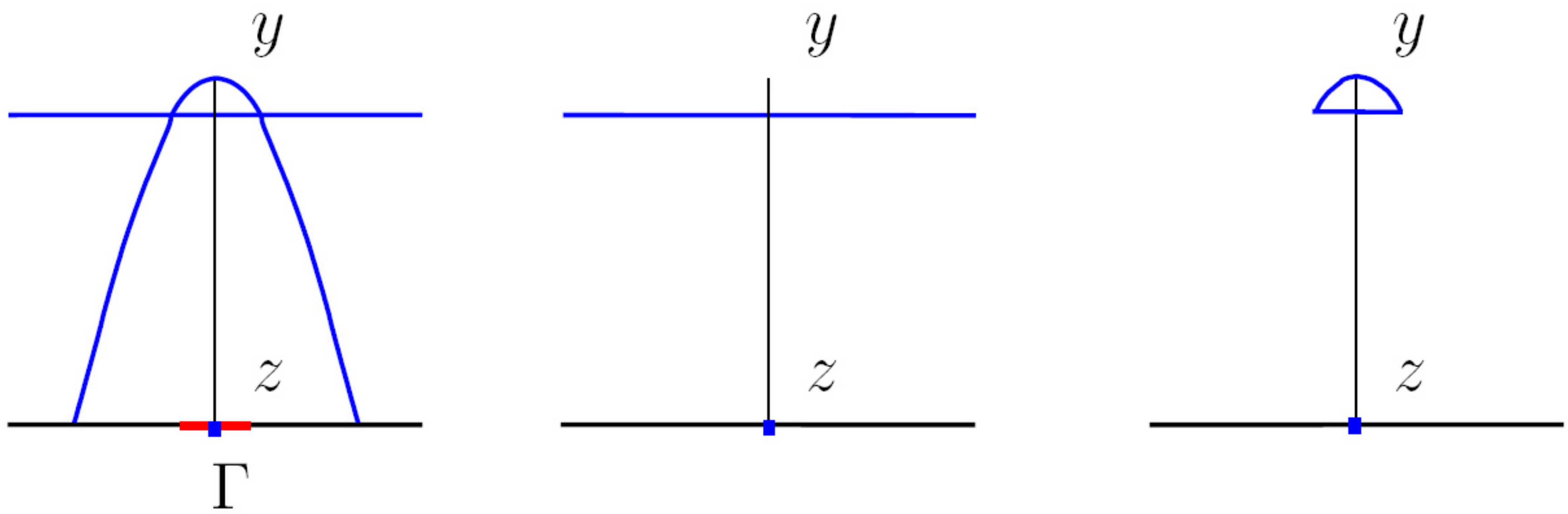


1. Send a probing signal
2. Measure and record the response
3. Send response back reversed in time



Time-reversal based focusing can recover sound speed inside unknown bodies

$$
\begin{aligned}
F &:= \alpha P(J\Lambda_{2T} - R\Lambda_{2T}RJ)f, \\
a_n &:= \Lambda_{2T}(h_n), \\
b_n &:= \Lambda_{2T}(RJh_n), \\
h_{n+1} &:= (1-\alpha^2)h_n + \alpha(PRb_n - PJa_n) + F
\end{aligned}
$$

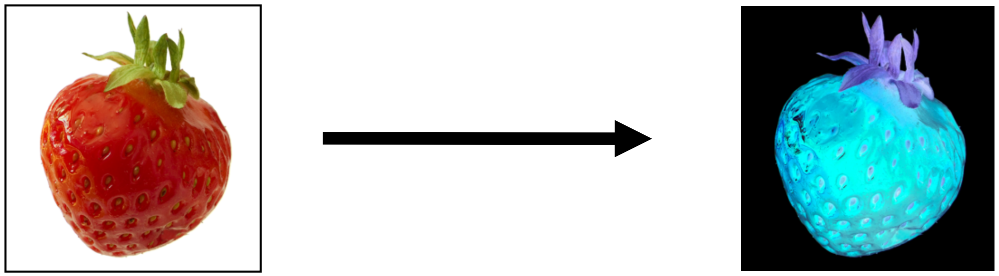


What are *not* inverse problems?

Example of a non-inverse problem: Inverting a photograph

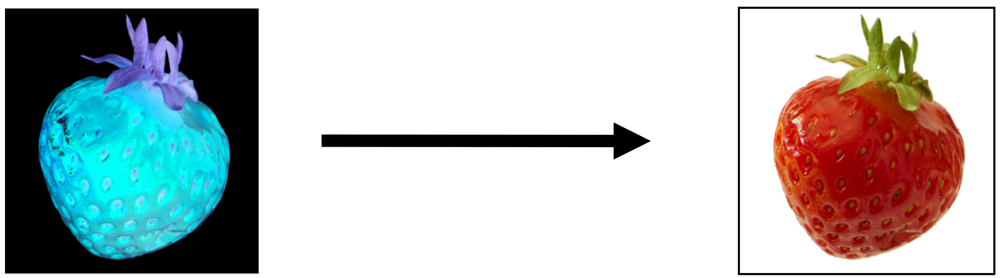
Direct problem:

Given a photograph, determine the negative image



"Inverse problem":

Given a negative, determine the positive image



Hadamard's definition of a "well-posed problem" has three parts

(H1) A solution exists

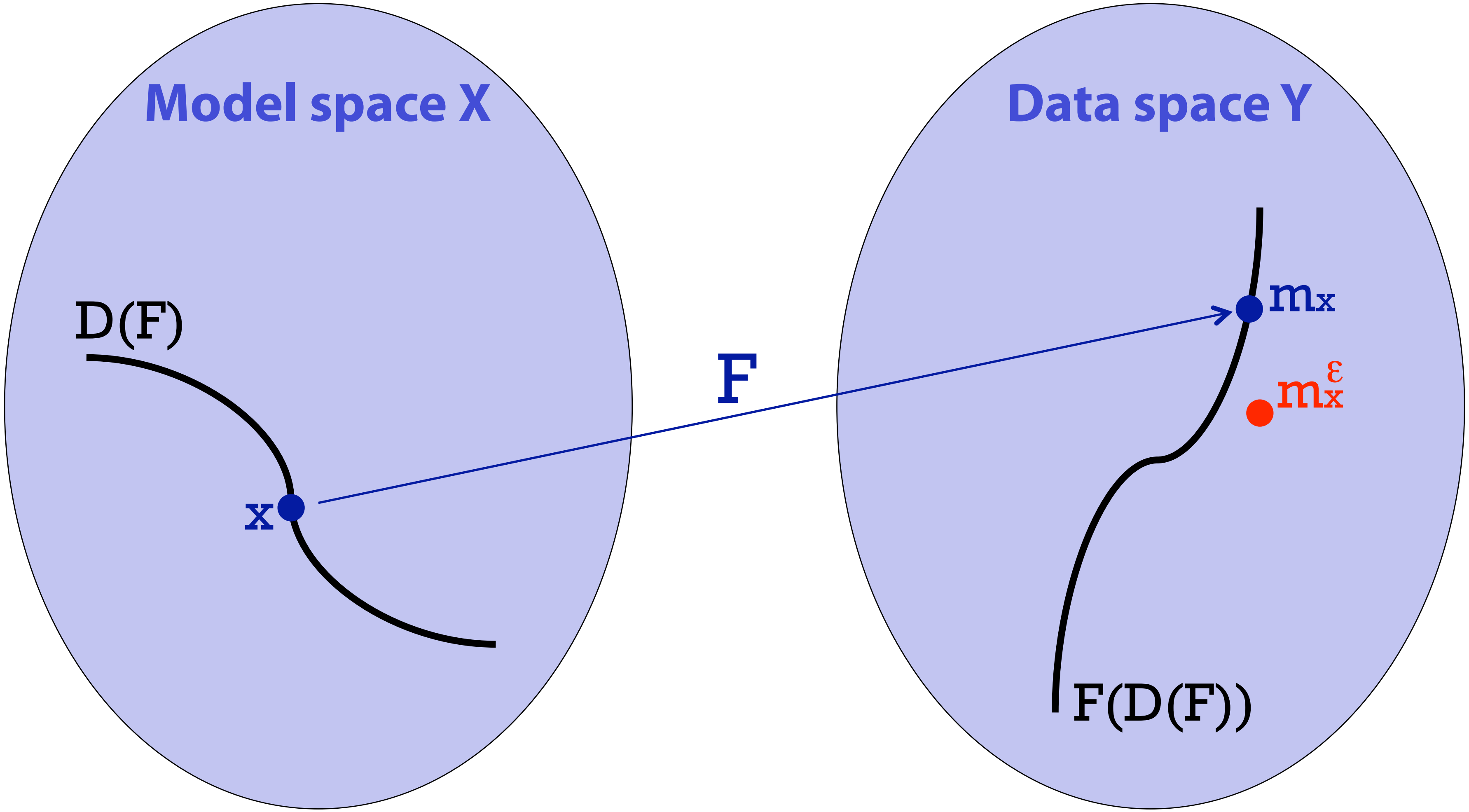
(H2) The solution is unique

(H3) The output depends continuously on the input

A problem is called "ill-posed", or inverse problem, if (H1), (H2) or (H3) fails.

Jacques Salomon Hadamard (1865-1963)

Ill-posedness in other words: the forward map F does not have a continuous inverse

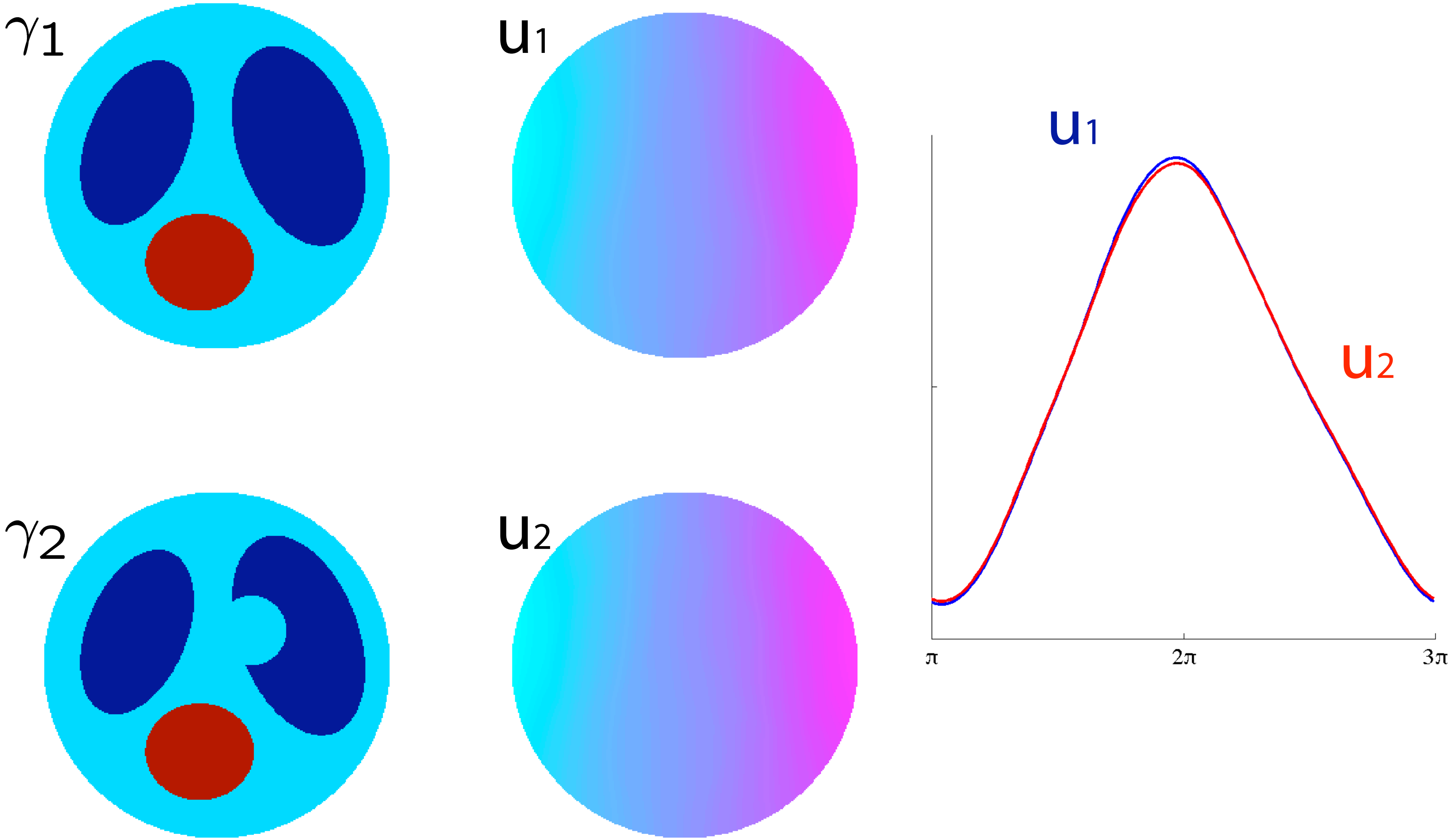


Example inverse problems revisited

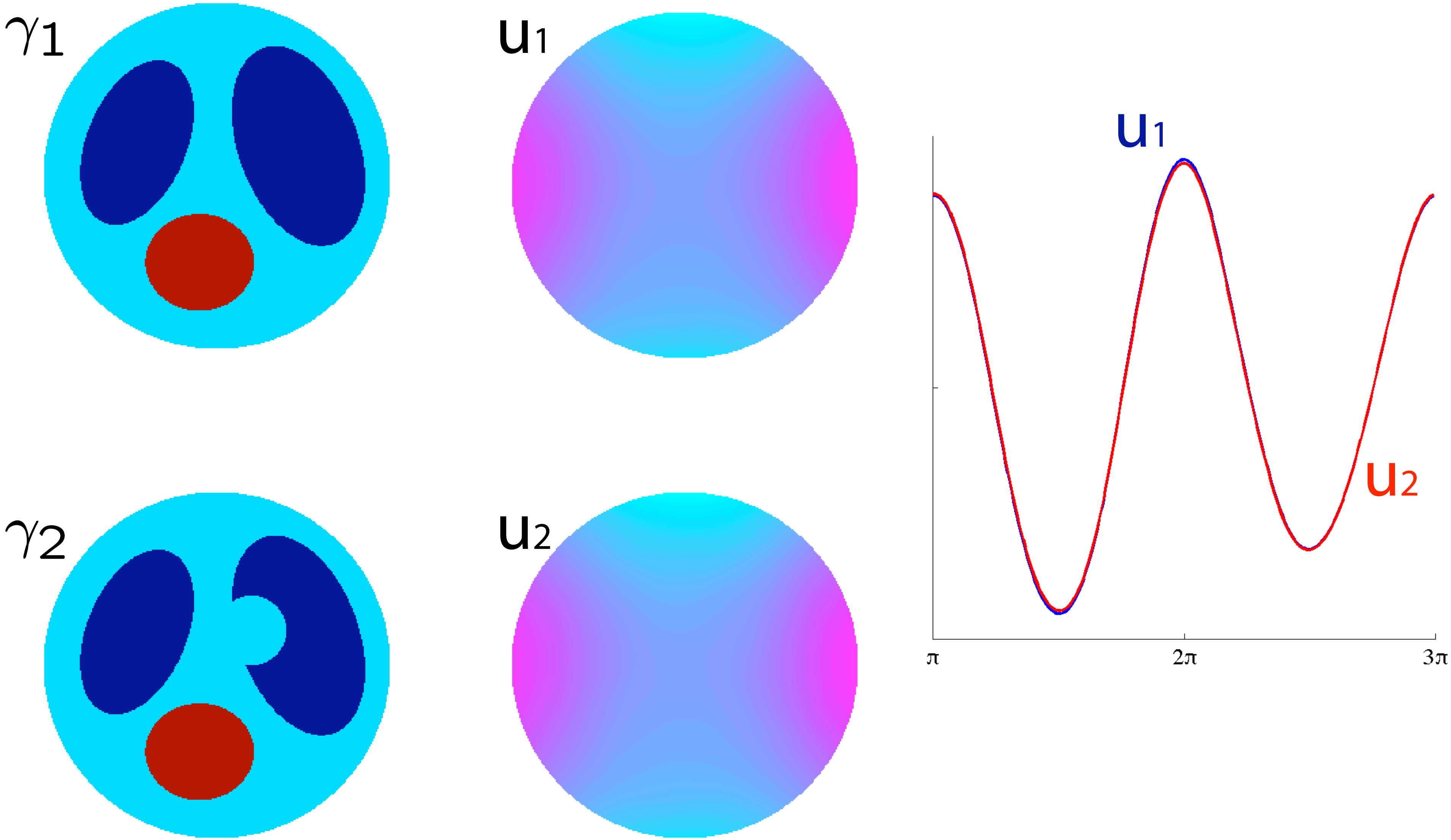
Image deblurring

Changing few pixel values on the left
changes the blurred image only slightly: (H3) fails

Electrical impedance tomography: choose two different conductivities



Electrical impedance tomography: big change in conductivity causes only small change in data

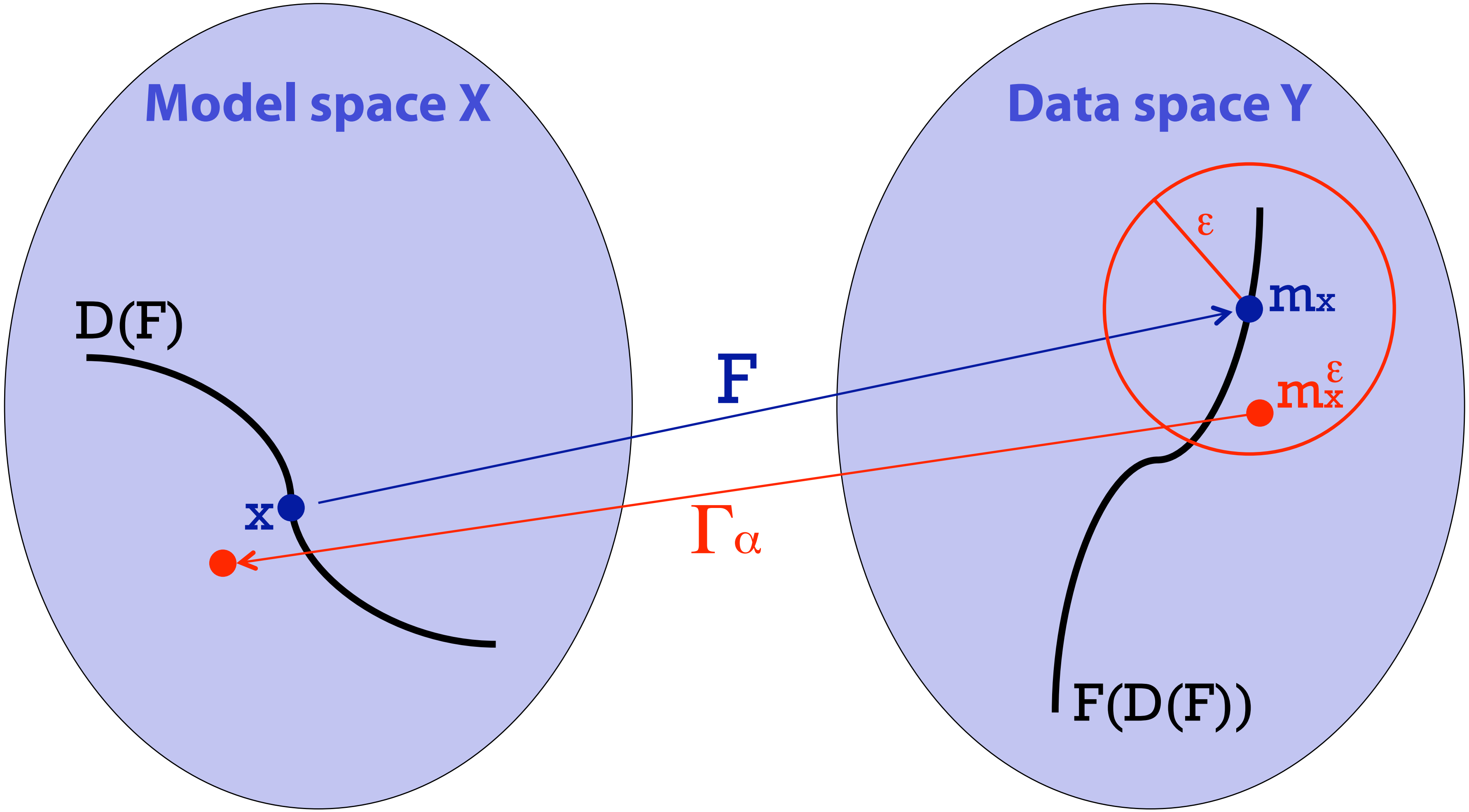


What is this course all about?

Goals of the course:

1. Learn how to write a practical inverse problem in matrix form: m=Ax+e
2. Learn how to detect ill-posedness from a matrix A using Singular Value Decomposition
3. Familiarize with two classes of solution methods: regularization and statistical inversion
4. Acquire skills to solve practical inverse problems using Matlab
5. Learn to report your scientific findings in writing

This is a schematic representation of the idea of regularization



Good to know

Matrix algebra

Least squares solution of linear systems

Basic Matlab programming (do you have access to Matlab and Image Processing Toolbox?)

Basic probability

How to pass the course?

Return solutions to exercise problems

Pass final exam

Complete project work