Inverse problems on Riemannian manifolds
Exercises, 26.04.2010

The exercises can be returned to Mikko Salo.

1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary, and assume that $\gamma = (\gamma^{jk})_{j,k=1}^n$ is a smooth positive definite symmetric matrix in $\overline{\Omega}$. If $F : \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism and if $F|_{\partial\Omega} = \mathrm{id}$, show that $\Lambda_{F_*\gamma} = \Lambda_\gamma$ where $F_*\gamma$ is the pushforward of γ, defined by
$$F_*\gamma(\tilde{x}) = \left.\frac{(DF)\gamma(DF)^t}{|\det(DF)|}\right|_{F^{-1}(\tilde{x})}$$
where $DF = (\partial_k F_j)_{j,k=1}^n$ is the Jacobian matrix.

2. If c is a smooth positive function, show that
$$\Delta_{cg} u = c^{-\frac{n+2}{4}}(\Delta_g + q)(c^{\frac{n-2}{4}} u)$$
where $q = c^{\frac{n-2}{4}} \Delta_{cg}(c^{-\frac{n-2}{4}})$.

3. Let (M, g) be a compact Riemannian manifold with boundary, and let $V \in L^\infty(M)$ be real valued. Show that there exist numbers $\lambda_1 \leq \lambda_2 \leq \ldots \to \infty$ such that $-\Delta_g + V - \lambda : H_0^1(M) \to H^{-1}(M)$ is an isomorphism whenever $\lambda \notin \{\lambda_1, \lambda_2, \ldots\}$, and further that there is an orthonormal basis $\{\phi_j\}_{j=1}^\infty$ of $L^2(M)$ such that $(-\Delta_g + V)\phi_j = \lambda_j \phi_j$ and $\phi_j \in H_0^1(M)$.

4. If $p_0 \in S^{n+1}$, prove that the stereographic projection is a conformal transformation from $S^{n+1} \setminus \{p_0\}$ onto $\mathbb{R}^n$ when these manifolds are equipped with their usual Riemannian metrics.

5. Let φ be a smooth function in (M, g). If $A := -h^2\Delta_g - |d\varphi|^2$ and $B := \frac{h}{i}(2\langle d\varphi, d\,\cdot\rangle + \Delta_g \varphi)$, check that A and B are formally self-adjoint.

6. If $(M, g) = (\Omega, e)$ and $0 \notin \overline{\Omega}$, verify that $\varphi(x) = \log|x|$ and $\varphi(x) = \frac{\alpha\cdot x}{|x|^2}$ are limiting Carleman weights. Here $\alpha \in \mathbb{R}^n$ is a fixed vector.

7. Prove the lemma in the lectures concerning the existence of semigeodesic coordinates.