(1) $dK_t = \alpha \cdot dt + dB_t$, $K_0 = 0$; $X_t = \cos K_t$, $Y_t = \sin K_t$

a) direct use of Ito-chain rule:

$$dX_t = (-\sin K_t)\, dK_t + \frac{1}{2}(-\cos K_t)\,\overbrace{(dK_t)^2}^{=dt}$$
$$= -Y_t \cdot (\alpha\, dt + dB_t) - \frac{1}{2} X_t\, dt$$
$$= (-\alpha Y_t - \frac{1}{2} X_t)\, dt - Y_t\, dB_t$$

- similarly:
$$dY_t = (\alpha X_t - \frac{1}{2} Y_t)\, dt + X_t\, dB_t$$

so
$$\begin{bmatrix} dX_t \\ dY_t \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{2} & -\alpha \\ \alpha & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} X_t \\ Y_t \end{bmatrix}}_{\equiv b(X_t, Y_t)} dt + \underbrace{\begin{bmatrix} -Y_t & 0 \\ X_t & 0 \end{bmatrix}}_{\equiv \sigma(X_t, Y_t)} \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix}$$

$W_t^1 := B_t$
$(W_t^2 : t \geq 0)$ independent copy of B_t.

b) Markov semigroup.
$$\mathcal{L} = \sum_{i,j} \left\{ b^i \frac{\partial}{\partial r^i} + \frac{1}{2} (\sigma\sigma^T)^{ij} \frac{\partial^2}{\partial r^i \partial r^j} \right\} \quad \text{where } \begin{cases} r^1 = x \\ r^2 = y \end{cases}$$

- in this case direct substitution gives
$$\mathcal{L} = (-\alpha y - \frac{1}{2} x) \frac{\partial}{\partial x} + (\alpha x - \frac{1}{2} y) \frac{\partial}{\partial y} + \frac{1}{2} \left\{ y^2 \frac{\partial^2}{\partial x^2} - 2xy \frac{\partial^2}{\partial x \partial y} + x^2 \frac{\partial^2}{\partial y^2} \right\}$$

- obviously, this representation can hold only for functions M that have second derivatives.

c) - Since K_t determines both X_t, Y_t uniquely it's easiest to work through (K_t) process.

- First, it's easy to see that $V_K(d\theta) = \frac{1}{2\pi} d\theta$ is the stationary measure for (K_t).

- To get stationary measure for (X_t, Y_t) we make a change-of-variables: $\forall f : \mathbb{R}^2 \to \mathbb{R}$ s.t. integrals converge:

$$\int_{\mathbb{R}^2} f(x,y)\, V(dx, dy) = \int_0^{2\pi} f(\cos\theta, \sin\theta)\, V_K(d\theta)$$
$$= \frac{1}{2\pi} \int_0^{\pi} \left[f(\cos\theta, +\sin\theta) + f(\cos\theta, -\sin\theta) \right] d\theta$$
$$= \frac{1}{2\pi} \int_{-1}^{1} \left[f(x, \sqrt{1-x^2}) + f(x, -\sqrt{1-x^2}) \right] \cdot \frac{1}{\sqrt{1-x^2}}\, dx$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \underbrace{\left\{ \frac{1}{2\pi} \mathbb{1}_{\{\pm\sqrt{1-x^2}\}}(dy) \cdot \frac{\mathbb{1}_{[-1,1]}(x)}{\sqrt{1-x^2}}\, dx \right\}}_{\equiv V(dx\, dy)}$$

- From the exercises we know that detailed balance holds iff $\int u (\mathcal{L} v)\, dV = \int (\mathcal{L} u) \cdot v\, dV$ for the stationary measure V.

- Clearly, (K_t) satisfies DB iff $\alpha = 0$: $\begin{cases} \mathcal{L}_K = -\alpha \frac{\partial}{\partial \theta} + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \\ \mathcal{L}_K^* = \alpha \frac{\partial}{\partial \theta} + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \end{cases}$

- This implies DB for (X_t, Y_t)-process:
  - By def. for $(x,y) \in S \equiv \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$:

$$\mathcal{L} M(x,y) = (\mathcal{L}_K (M \circ f))(f^{-1}(x,y))$$
where $f : [0, 2\pi] \to \mathbb{R}^2$, $f(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

1c) continues... Let $Q=(x,y)$

$$\int v(Q)\,\mathcal{A}M(Q)\,\nu(dQ) = \int v(Q)\left[\mathcal{A}_K(M\circ f)\right](f^{-1}(Q))\,\nu(dQ)$$

$$= \int (v\circ f)(\theta)\left[\mathcal{A}_K(M\circ f)\right](\theta)\,\nu_K(d\theta)$$

$$\overset{\text{DB of } (K_t)}{=} \int \left[\mathcal{A}_K(v\circ f)\right](\theta)\cdot(M\circ f)(\theta)\,\nu_K(d\theta)$$

$$= \int \left[\mathcal{A}_K(v\circ f)\right](f^{-1}(Q))\,M(Q)\,\nu(dQ)$$

$$= \int \mathcal{A}v(Q)\cdot M(Q)\,\nu(dQ) \qquad \blacksquare$$

1d) Nothing changes!

- This can be seen by showing that $\mathcal{A}_K$ and hence $\mathcal{A}$ remain intact:

▷ define: $M_t := \int_0^t X_s\,dB_s - \int_0^t Y_s\,dB'_s \quad (t\geq 0)$

- It's easy to see that (M_t) is BM. This follows by showing that $\mathcal{A}_M = \frac{1}{2}\frac{\partial^2}{\partial\theta^2}$:

Indeed: $$\mathcal{A} = 0\cdot\frac{\partial}{\partial\theta} + \frac{1}{2}\left(\cos^2\theta\,\frac{\partial^2}{\partial\theta^2} + (-\sin\theta)^2\frac{\partial^2}{\partial\theta^2}\right) = \frac{1}{2}\frac{\partial^2}{\partial\theta^2}$$

<u>NOTE</u>: If one calculates directly from the definitions one gets:

$$\begin{bmatrix} dX_t \\ dY_t \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{2} & -\alpha \\ \alpha & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} X_t \\ Y_t \end{bmatrix}}_{=b}dt + \underbrace{\begin{bmatrix} -X_tY_t & Y_t^2 \\ X_t^2 & -X_tY_t \end{bmatrix}}_{=\tilde{\sigma}}\begin{bmatrix} dB_t \\ dB'_t \end{bmatrix}$$

same as before!

$\tilde{\sigma}$ ← different

- Even though $\tilde{\sigma}$ is different $\mathcal{A}$ remains the same as it should:

$$\mathcal{A} = \sum_i b^i\frac{\partial}{\partial t_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)^{ij}\frac{\partial^2}{\partial t_i\partial t_j}$$

since $$(\sigma\sigma^T)(x,y) = \begin{bmatrix} (xy)^2+y^4 & -x^3y-xy^3 \\ -x^3y-xy^3 & x^4+(xy)^2 \end{bmatrix} = \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix}$$

$$-x^3y - xy^3 = -xy\overbrace{(x^2+y^2)}^{=1}, \text{ etc}$$

- The "−" sign plays also no role:

$$d\theta_t = \alpha\,dt + x_t\,dB_t - y_t\,dB'_t$$

$\tilde{B}_t := -B'_t$ is also BM independent of B_t!

(2) $H(P,Q) = \sum_{x=0}^{\cdots} \left\{ \frac{P_x^2}{2} + U(Q_x) + V(Q_{x+1}-Q_x) \right\}$ with $Q_0 = Q_{M+1} := 0$

- By Ito calculus $\mathcal{E}(t) := H(P(t), Q(t))$ satisfies:

$$d\mathcal{E} = \sum_x \left\{ \frac{\partial H}{\partial P_x} dP_x + \frac{\partial H}{\partial Q_x} dQ_x + \frac{1}{2} \frac{\partial^2 H}{\partial P_x^2} (dP_x)^2 \right\}$$

- NOTE: that terms "$dP_x dQ_x$", "$(dQ_x)^2$" are absent since $dQ_x = \frac{\partial H}{\partial P_x} dt$ does not contain stochastic terms dW_x.
- NOTE: terms "$dP_x dP_{x'}$" are also absent since $(w_1(t) : t \geq 0)$ and $(w_M(t) : t \geq 0)$ are independent while $dP_x = -\frac{\partial H}{\partial Q_x} dt$ for $x \neq 1, M$ has no stochastic terms.

- Thus:

$$d\mathcal{E} = \sum_x \underbrace{\left\{ \frac{\partial H}{\partial P_x}\left(-\frac{\partial H}{\partial Q_x} dt\right) + \frac{\partial H}{\partial Q_x}\left(\frac{\partial H}{\partial P_x} dt\right) \right\}}_{=0}$$

$$+ \left\{ \frac{\partial H}{\partial P_1}\left(dP_1 + \frac{\partial H}{\partial Q_1} dt\right) + \frac{1}{2}\frac{\partial^2 H}{\partial P_1^2}(dP_1)^2 \right\} + \left\{ \text{like prev. term but } 1 \mapsto M \right\}$$

- Now, $\frac{\partial H}{\partial P_x} = P_x$ and $\frac{\partial^2 H}{\partial P_x^2} = 1$ and $\left(dP_1 + \frac{\partial H}{\partial Q_1} dt\right) = -\gamma P_1 \cdot dt + \sqrt{2\gamma T_1} \cdot dw_1$, and

$$(dP_x)^2 = 2\gamma T_1 \cdot dt$$

so we get:

$$d\mathcal{E} = \left\{ P_1 \cdot \left(-\gamma P_1 \cdot dt + \sqrt{2\gamma T_1}\, dw_1\right) + \frac{1}{2} \cdot 1 \cdot (2\gamma T_1 \cdot dt) \right\} + \left\{ 1 \mapsto M \right\}$$

$$= \gamma \cdot (T_1 - P_1^2)\, dt - \gamma \cdot (P_M^2 - T_M)\, dt + \sqrt{2\gamma T_1} \cdot P_1\, dw_1 + \sqrt{2\gamma T_M} \cdot P_M \cdot dw_M$$

$\Rightarrow$ we identify:

$$j_1(P_1) = \gamma \cdot (T_1 - P_1^2) \qquad \text{and} \qquad \eta_1(P,Q) = \sqrt{2\gamma T_1} \cdot P_1$$
$$j_M(P_M) = \gamma\,(P_M^2 - T_M) \qquad \eta_M(P,Q) = \sqrt{2\gamma T_M} \cdot P_M$$

- This is what was asked in part (a).

(b) Since Z_T is a constant, it suffices to show:

$$\mathcal{L}^*(e^{-\beta H}) = 0, \quad \beta = \frac{1}{T}$$

$\forall \phi$ is smooth with compact support.

where $\int_{(\mathbb{R}\times\mathbb{R})^M} \rho(P,Q)\,(\mathcal{L}\phi)(P,Q)\, dP\,dQ = \int_{(\mathbb{R}\times\mathbb{R})^M} (\mathcal{L}^*\rho)(P,Q)\, \phi(P,Q)\, dP\,dQ$

i.e., $\mathcal{L}^*$ is Leb-adjoint of the Markov semigroup generator $\mathcal{L}$ of the process $\{(P(t), Q(t)) : t \geq 0\}$

- Let's first compute $\mathcal{L}$:

$$\mathcal{L} = \sum_x \left\{ b^{P_x} \frac{\partial}{\partial P_x} + b^{Q_x}\frac{\partial}{\partial Q_x} \right\} + \frac{1}{2}(\sigma^{P_1})^2 \frac{\partial^2}{\partial P_1^2} + \frac{1}{2}(\sigma^{P_M})^2 \frac{\partial^2}{\partial P_M^2}$$

$$= \sum_x \left\{ \left(-\frac{\partial H}{\partial Q_x}\right)\frac{\partial}{\partial P_x} + \left(\frac{\partial H}{\partial P_x}\right)\frac{\partial}{\partial Q_x} \right\} + \gamma \cdot \left\{ -P_1 \frac{\partial}{\partial P_1} + T_1 \cdot \frac{\partial^2}{\partial P_1^2} \right\}$$
$$+ \gamma \left\{ -P_M \frac{\partial}{\partial P_M} + T_M \frac{\partial^2}{\partial P_M^2} \right\}$$

$$=: \bar{\mathcal{L}} + \gamma \cdot \mathcal{L}_1 + \gamma \cdot \mathcal{L}_M$$

- We'll show that: $\mathcal{L}_1^* e^{-\beta H} = \mathcal{L}_M^* e^{-\beta H} = \bar{\mathcal{L}}^* e^{-\beta H} = 0$.
- First, we note that $\bar{\mathcal{L}}^* e^{-\beta H} = 0$ as $\bar{\mathcal{L}}$ describes pure Hamiltonian evolution which conserves energy.

2b continuess...

- Let's then consider $\mathcal{L}_1^*$:

$$\int \rho\,(\mathcal{L}_1\phi)\,dp\,dQ = \int (\mathcal{L}_1^*\rho)\cdot\phi\,dp\,dQ$$ $\forall$ prod. densities ρ & $\forall\,\phi$ smooth compact support.

- Since ϕ has compact support partial integration gives:

$$\mathcal{L}_1^*\rho = \frac{\partial}{\partial P_1}(P_1\rho) + T_1\frac{\partial^2}{\partial P_1^2}$$

- If $\rho = e^{-\frac{1}{T}H}$ this gives:

$$\mathcal{L}_1^* e^{-\beta H} = e^{-\beta H}\cdot\left\{1 + P_1\Big(-\beta \overbrace{\frac{\partial H}{\partial P_1}}^{P_1}\Big)\right\} + \frac{1}{\beta}\frac{\partial}{\partial P_1}\left\{\Big(-\beta \overbrace{\frac{\partial H}{\partial P_1}}^{P_1}\Big)e^{-\beta H}\right\}$$

$$= e^{-\beta H}\left\{1-\frac{P_1^2}{T}\right\} - e^{-\beta H}\left\{1-\frac{P_1^2}{T}\right\} = 0$$

- $\mathcal{L}_M^* e^{-\beta H} = 0$ comes in the same way.

2c) $$\int \dot{j}_1(P,Q)\,N_T(dp\,dQ) = \lambda\cdot\left(T - \int_{(\mathbb{R}\times\mathbb{R})^M} P_1^2\,N_T(dp\,dQ)\right) \qquad (A)$$

Now, the marginal measure $\tilde{N}_T$ defined by:

$$\tilde{N}_T(B) := \int_{B\times\mathbb{R}^M} N_T(dp\,dQ), \quad \forall\, B \subset \mathrm{Borel}(\mathbb{R}^M)$$

is Gaussian:

$$\tilde{N}_T(dP) = \prod_x \frac{1}{\sqrt{2\pi T}}\, e^{-\frac{1}{2}\left(\frac{P_x}{\sqrt{T}}\right)^2}$$

And therefore

$$\int P_1^2 N_T(dp\,dQ) = \int P_1^2\,\tilde{N}_T(dP) = T$$

which yields: $\int \dot{j}_1(P,Q)\,N_T(dp\,dQ) = 0$.

2d) By def. $\mathcal{L}^* N_{T_1,T_M} = 0$

Therefore: $$\int (\mathcal{L}H)(P,Q)\,N_{T_1,T_M}(dP,dQ) = \int H(P,Q)\,\overbrace{\mathcal{L}^* N_{T_1,T_M}(dP,dQ)}^{=0} = 0 \qquad (**)$$

- Note: here $\mathcal{L}^*$ denotes an operator acting on measures and is therefore not equal to the Lebesgue-adjoint of $\mathcal{L}$ denoted in 2b also by $\mathcal{L}^*$.

But $$\mathcal{L}H(P,Q) = \lim_{t\to 0}\frac{1}{t}\left\{\mathbb{E}\left[H(P_t,Q_t)\mid (P_0,Q_0)=(P,Q)\right] - H(P,Q)\right\}$$

$$= \lim_{t\to 0}\frac{1}{t}\,\mathbb{E}\left[\int_0^t d\mathcal{E}(t)\mid (P_0,Q_0)=(P,Q)\right]$$

$$= \lim_{t\to 0}\frac{1}{t}\,\mathbb{E}\left[\int_0^t \dot{j}_1\,dt - \int_0^t \dot{j}_M\,dt + \int_0^t \gamma_1\,dw_1 + \int_0^t \gamma_M\,dw_M\right]$$

- Since $\mathbb{E}\left[\int_0^t f(s)\,dw_x(s)\right] = 0$ for any process $(f(s): s\geq 0)$ adapted to the filtration $\mathbb{F} = (\mathcal{F}_s : s\geq 0)$, generated by w_1, w_M we get using $\frac{1}{t}\int_0^t \dot{j}_x(P_x(s))\,dt \to \dot{j}_x(P_x)$:

$$\mathcal{L}H(P,Q) = \dot{j}_1(P_1) - \dot{j}_M(P_M)$$

- Substituting this into (**) completes the proof.
- $\bar{J}(T_1,T_M)$ = "average energy flow through the chain at the steady state".

(3) We have: $\Phi(f) = \frac{1}{\sqrt{2}}\{a^{\dagger}(f) + a(f)\}$, $W(f) := e^{i\Phi(f)}$

$$E_\Omega(f) := \langle \Omega, W(f)\Omega \rangle = \sum_{l=0}^{\infty} \frac{i^l}{l!} \langle \Omega, \Phi^l(f)\Omega \rangle$$

For $l = 2m+1$, $\langle \Omega, \Phi^l(f)\Omega \rangle = 0$ since each term corresponding the expansion of $\Phi(f)^l$ has different number of $a^\dagger(f)$ and $a(f)$ operations in the Normal form, i.e., in the form where one has commuted $a(f)$ to the right of $a^\dagger(f)$:s, and

$$\langle \Omega, (a(f))^{m_1} (a^\dagger(f))^{m_2} \Omega \rangle = \langle \underbrace{a^\dagger(f)^{m_1}\Omega}_{\text{"}m_1\text{-particle space"}}, \underbrace{a^\dagger(f)^{m_2}\Omega}_{\text{"}m_2\text{-particle space"}} \rangle = 0$$

ORTOGONAL

So, one is left with

$$E_\Omega(f) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \langle \Omega, \Phi^{2m}(f)\Omega \rangle$$

drop f's to make notation lighter.

$$\text{Now, } \langle \Omega, \Phi^{2m}(f)\Omega \rangle = \frac{1}{\sqrt{2}} \langle \Omega, (a^\dagger + a)\Phi^{2m-1}\Omega \rangle$$

$$= \underbrace{\frac{1}{\sqrt{2}} \langle \underbrace{a\Omega}_{=0}, \Phi^{2m-1}\Omega \rangle}_{=0} + \frac{1}{\sqrt{2}} \langle \Omega, a \cdot \Phi^{2m-1}\Omega \rangle$$

$$\text{Here } a\Phi = \frac{1}{\sqrt{2}} a(a^\dagger + a) = \frac{1}{\sqrt{2}}\{[a, a^\dagger] + a^\dagger a + a^2\} = \frac{1}{\sqrt{2}}\|f\|^2 + \Phi \cdot a$$

$$\text{so } \langle \Omega, \Phi^{2m}(f)\Omega \rangle = \frac{\|f\|^2}{2} \langle \Omega, \Phi^{2(m-1)}\Omega \rangle + \frac{1}{\sqrt{2}} \langle \Omega, \Phi \cdot a \cdot \Phi^{2m-2}\Omega \rangle$$

$$= \ldots \text{"moving } a \text{ to right"} \ldots$$

$$= \frac{2m-1}{2} \|f\|^2 \cdot \langle \Omega, \Phi^{2(m-1)}\Omega \rangle \quad (*)$$

where $(2m-1)$ is the number of terms a has to pass and create the commutator $\|f\|^2$.

Induction on $(*)$ gives

$$\langle \Omega, \Phi^{2m}\Omega \rangle = \left(\frac{\|f\|^2}{2}\right)^m \frac{(2m)!}{2^m m!}$$

- Substituting this into the series yields:

$$E_\Omega(f) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \frac{(2m)!}{2^m m!} \left(\frac{\|f\|^2}{2}\right)^m = \underline{e^{-\frac{\|f\|^2}{4}}}$$

3b) WE MUST COMPUTE

$$E_{V,\theta_0}(f) = \langle \varphi, W(f)\varphi \rangle$$

WHERE $\varphi := \frac{1}{\sqrt{M_0!}} a^\dagger(\phi_0)^{M_0}\Omega$ AND $\phi_0(x) = \frac{1}{(2L)^{d/2}}$ IS THE GROUND STATE OF THE SINGLE PARTICLE HAMILTONIAN $h = -\Delta$ (IN COORD. BASIS) OF THE BOX $V = [-L,L]^d$.

- WE HAVE:

$$E_{V,\theta_0}(f) = \frac{1}{M!}\langle \Omega, a_0^M W a_0^{\dagger M}\Omega\rangle, \quad \begin{matrix} M \equiv M_0 \\ a_0 \equiv a(\phi_0),\ W \equiv W(f),\ a \equiv a(f) \end{matrix}$$

- THE IDEA IS AGAIN TO <u>COMMUTE a_0's FROM LEFT TO RIGHT:</u> TO ACHIEVE THIS, NOTICE:

$$a_0 \Phi^l = \frac{1}{2^{l/2}}\{a_0(a^\dagger + a)\}(a^\dagger+a)^{l-1} = \frac{1}{2^{l/2}}\{\langle\phi_0,f\rangle + (a^\dagger+a)a_0\}(a^\dagger+a)^{l-1}$$

$$= \frac{1}{\sqrt{2}}\langle\phi_0,f\rangle\Phi^{l-1} + \frac{1}{2^{l/2}}(a^\dagger+a)\{a_0(a^\dagger+a)\}(a^\dagger+a)^{l-2}$$

$$= \dots = \frac{l}{\sqrt{2}}\langle\phi_0,f\rangle\Phi^{l-1} + \Phi^l a_0$$

i.e., $[a_0, \Phi^l] = \frac{l}{\sqrt{2}}\langle\phi_0,f\rangle\Phi^{l-1}$

- THIS IN TURN IMPLIES:

$$[a_0, W] = \sum_{l=0}^{\infty}\frac{i^l}{l!}[a_0,\Phi^l] = \frac{i}{\sqrt{2}}\langle\phi_0,f\rangle W$$

- Pulling a^M past W IS NOW GIVEN BY:

$$a(\phi_0)^M W(f) = \sum_{j=0}^{M}\binom{M}{j}\cdot\left(i\frac{\langle\phi_0,f\rangle}{\sqrt{2}}\right)^{M-j} W(f)\,a(\phi_0)^j \quad (1)$$

- THIS follows BY INDUCTION. AlternativeLY ONE obtains this by NOTICING that j out of M $a(\phi_0)$ TERMS are COMMUTED THROUGH $W(f)$ while the remaining $M-j$ UNDERGO CONTRACTION with $W(f)$.

- USING (1) WE GET:

$$E_{V,\theta_0}(f) = \frac{1}{M!}\sum_{j=0}^{M}\binom{M}{j}\left\{i\frac{\langle\phi_0,f\rangle}{\sqrt{2}}\right\}^{M-j}\langle\Omega, W(f)a(\phi_0)^j a^\dagger(\phi_0)^M\Omega\rangle \quad (2)$$

- AGAIN, WE'll COMMUTE a^j's OVER $a^{\dagger M}$'s: FIRST: | $a \equiv a_0 \equiv a(\phi_0)$

$$a(a^\dagger)^M\Omega = (a a^\dagger)a^{\dagger M-1}\Omega = (\|\phi_0\|^2 + a^\dagger a)a^{\dagger(M-1)}\Omega$$

$$= \|\phi_0\|^2\cdot(a^\dagger)^{M-1}\Omega + a^\dagger\{a(a^\dagger)^{M-1}\Omega\}$$

$$= \dots = M\cdot\underbrace{\|\phi_0\|^2}_{=1}(a^\dagger)^{M-1}\Omega$$

- THUS

$$a^j(a^\dagger)^M\Omega = \frac{M!}{(M-j)!}(a^\dagger)^{M-j}\Omega$$

- USING THIS IN (2) ONE obtains:

$$E_{V,\theta_0}(f) = \sum_{j=0}^{M}\binom{M}{j}\frac{1}{(M-j)!}\left\{i\frac{\langle\phi_0,f\rangle}{\sqrt{2}}\right\}^{M-j}\langle\Omega, W(f)(a^\dagger(\phi_0))^{M-j}\Omega\rangle \quad (3)$$

- ONCE AGAIN WE'll COMMUTE $a^\dagger(\phi_0)$ past $W(f)$ TO THE left.
- WE don't have TO start from SCRATCH HOWEVER, SINCE THE REQUIRED COMMUTATION Rule follows from taking adjoints of (1):

3b continues....

Indeed, since $W(f)^\dagger = W(-f)$ we get

$$W(f)\,a^\dagger(\phi_0)^{M-j} = \left\{ a(\phi_0)^{M-j}\, W(-f) \right\}^\dagger \overset{(1)}{=} \sum_{\ell=0}^{M-j} \binom{M-j}{\ell} \left\{ -i \frac{\langle \phi_0, -f\rangle}{\sqrt{2}} \right\}^{M-j-\ell} a^\dagger(\phi_0)^\ell\, W(f) \qquad (4)$$

Using this in (3) gives

$$E_{V,\theta_0}(f) = \left(\sum_{j=0}^{M} \binom{M}{j} \frac{1}{(M-j)!} \left\{ -\frac{1}{2} |\langle \phi_0, f\rangle|^2 \right\}^{M-j} \right) \underbrace{\langle \Omega, W(f)\Omega\rangle}_{= E_\Omega(f)} \qquad (5)$$

since only the $(\ell=0)$-term in (4) survives: $\langle \Omega, a^\dagger(\phi_0)^\ell \cdots \Omega\rangle = 0$ unless $\ell = 0$

- This is the first formula we were asked to establish.

- To obtain the limit we note that $\phi_0(\cdot) = \frac{1}{|V|^{1/2}} = \text{const.}$ and thus

$$\frac{1}{2} |\langle \phi_0, f\rangle|^2 = \frac{1}{2} \left| |V|^{-1/2} \int_V f(x)\,dx \right|^2 = \frac{(2\pi)^d}{2|V|} \left| (2\pi)^{-\frac{d}{2}} \int_V f(x)\,dx \right|^2$$

$$\overset{(*)}{=} \frac{(2\pi)^d}{2} \left(\frac{\theta_0}{M}\right) |\hat{f}(0)|^2 = \frac{1}{M} \frac{(2\pi)^d}{2} |\hat{f}(0)|^2 \cdot \theta_0 \qquad (6)$$

where $(*)$ holds because $\mathrm{spt}(f)$ is compact and therefore bounded so that $\int_V f\,dx = \int_{V\cap \mathrm{spt}(f)} f\,dx$ equals $\int_{\mathrm{spt}(f)} f\,dx = \int_{\mathbb{R}^d} f\,dx$, provided V is large enough.

- Now substituting (6) into (5) and taking the limit $(M\to\infty)$ yields:

$$E_{\theta_0}(f) = E_\Omega(f) \cdot j_0\left((2\pi)^{d/2} (2\theta_0)^{1/2} \cdot |\hat{f}(0)| \right)$$

c) Excited states have non-zero density:

- Since 1-particle Hamiltonian is self-adjoint: $\langle \phi_i, \phi_j\rangle = \delta_{ij}$

$$\Rightarrow \left[a^\#(\phi_i), a^\times(\phi_j) \right] = 0 \quad \text{unless } i=j \text{ and } \{\#, \times\} = \{+, (\cdot)\}.$$

(empty.)

- In the part (b) the only properties we used were
  - (i) commutation relations between $a(\phi_0)$, $a^\dagger(\phi_0)$ and $W(f)$
  - (ii) $a(\phi_0)\Omega = 0$
  - (iii) $\langle \phi_0, f\rangle = (2\pi)^{d/2} |V|^{1/2} \hat{f}(0)$

- Now, (i) & (ii) remain intact if one replaces

$$\phi_0 \mapsto \phi_j \quad \text{and} \quad \Omega \mapsto \left(\prod_{i\in I} a^\dagger(\phi_i)^{M_i} \right)\Omega, \quad j \notin I, \quad M_i \in \mathbb{N}_0, \ I \subset \mathbb{N}_0$$

- Since different excitations do not see each others one gets for any $i \in \mathbb{N}_0$:

$$E_{V,\underline{\theta}}(f) = \sum_{j=0}^{M_i} \binom{M}{j} \frac{1}{(M_i-j)!} \left\{ -\frac{1}{2} |\langle \phi_i, f\rangle|^2 \right\}^{M_i-j} \langle \psi_{V,\underline{\theta}^{(i)}}, W(f)\psi_{V,\underline{\theta}^{(i)}}\rangle$$

where $\psi_{V,\underline{\theta}^{(i)}} = \left(\prod_{\substack{j=0 \\ j\neq i}}^{\ell} \frac{1}{\sqrt{M_j!}} a^\dagger(\phi_j)^{M_j} \right)\Omega$

- applying this repeatedly for each $j = 0, 1 \ldots, \ell$, in turn thus gives (just like (5) above)

$$E_{V,\underline{\theta}}(f) = E_\Omega(f) \cdot \prod_{k=0}^{\ell} \left\{ \sum_{j_k=0}^{M_k} \binom{M_k}{j_k} \frac{1}{(M_k-j_k)!} \left(-\frac{|\langle \phi_k, f\rangle|^2}{2} \right)^{M_k-j_k} \right\} \qquad (5')$$

3c continues..

- This brings us to $\langle \phi_j, f\rangle = ((2\pi)^d |V|)^{1/2} \hat{f}(k_j)$, $k_j \in \left(\frac{2\pi}{L}\mathbb{Z}\right)^d$

so that
$$\frac{|\langle \phi_j, f\rangle|^2}{2} = \frac{1}{M_j}\frac{(2\pi)^d}{2}|\hat{f}(0)|^2 \cdot \theta_j \qquad (6')$$

- continuing just like in (D) using (6') in (5') yields Eq. (0.4) for

$$E_{\underline{\theta}}(f) = \lim_{|V|\to\infty} E_{V,\underline{\theta}}(f).$$

d) continuous distribution of momentum:

- We're going to obtain $E_\theta(f)$ for a density function $\theta: \mathbb{R}^d \to \mathbb{R}_+$.
- To make connection between previously considered density vectors we discretize:

$\tilde{\theta} := (\tilde{\theta}_j : j \in \{0,1,\dots,m\}^d)$, $\tilde{\theta}_j := \theta(k_j)\Delta k$, $j \equiv (j^1 \dots j^d) \in \{0,1,\dots,m\}^d$

with $k_j \equiv (k_j^1, \dots, k_j^d) \in [-K,K]^d$, $k_j^i = -K + \frac{2K}{m} j^i$, $\Delta k = \left(\frac{2K}{m}\right)^d$ = "volume of disc. box"

where $K > 0$ and $m \in \mathbb{N}$. (In the exam paper $\underline{\theta}^{(K,m)} \equiv \tilde{\theta}$.)

- In other words, $\tilde{\theta}$ is a density vector obtained by discretizing θ inside a momentum box $[-K,K]^d$.
- To obtain general $E_\theta(f)$ we take limits:

i) $m \to \infty$

ii) $K \to \infty$

- From (0.4) we got:

$O(f)$ means any function g s.t. $\frac{g}{f} \to 0$ in the limit

$$\ln\left\{\frac{E_{\tilde{\theta}}(f)}{E_\Omega(f)}\right\} = \sum_{j \in \{0,\dots,m\}^d} \ln \circ J_0\left((2\pi)^{d/2}(2\tilde{\theta}_j)^{1/2}\cdot|f(k_j)|\right)$$

$$= \sum_j -\frac{1}{4}\left((2\pi)^{d/2}(2\tilde{\theta}_j)^{1/2}|f(k_j)|\right)^2 + O((\dots)^2)$$

Number of terms $= \# j$

$$= -\frac{(2\pi)^d}{2}\sum_j |f(k_j)|^2\tilde{\theta}_j + O(\dots)$$

$$= -\frac{(2\pi)^d}{2}\sum_j |f(k_j)|^2\theta(k_j)\Delta k + \underbrace{O\Big(m^d \cdot \{\overbrace{\left(\tfrac{2K}{m}\right)^d}^{\Delta k}\}\Big)}_{=o(1)\text{, i.e., } \to 0 \text{ as } m\to\infty.}$$

- Thus:

$$\lim_{m\to\infty} \ln\left\{\frac{E_{\underline{\theta}^{(K,m)}}(f)}{E_\Omega(f)}\right\} = -\frac{(2\pi)^d}{2}\int_{[-K,K]^d}\theta(k)|\hat{f}(k)|^2 dk$$

- Now, if f is such that $(k \mapsto \theta(k)|\hat{f}(k)|^2)$ is integrable: we can take the second limit $K \to \infty$ as well:

$$\ln\left\{\frac{E_\theta(f)}{E_\Omega(f)}\right\} = -\frac{(2\pi)^d}{2}\int_{\mathbb{R}^d}\theta|\hat{f}|^2 dk$$

or in other words, recalling $E_\Omega(f) = e^{-\frac{1}{4}\langle \hat{f}, \hat{f}\rangle}$ one obtains

$$E_\theta(f) = \exp\left[-\frac{1}{4}\langle \hat{f}, (1 + 2(2\pi)^d\theta)\hat{f}\rangle\right]$$

3e) IN THIS CASE THE DENSITY IS OF THE FORM

$$\tilde{\theta}(k) = \theta(k) + \theta_0 \cdot \delta(k)$$

BY EXCLUDING $k=0$ FROM THE DISCRETIZATION PROCESS OF PART d) ONE OBTAINS

$$E_{\theta_0,\theta}(f) = E_{\Omega}(f) \cdot j_0\left((2\pi)^{d/2}\sqrt{2\theta_0}\cdot|\hat{f}(0)|\right)\cdot \exp\left[-\frac{(2\pi)^d}{2}\int_{\mathbb{R}^d}\theta(k)\,|\hat{f}(k)|^2\,dk\right]$$

THE DYNAMICS OF THE HAMILTONIAN COMMUTE WITH THE MOMENTUM:

$$W(f) \xmapsto{t} W(e^{it\omega}\cdot f)$$

WITH $\omega(k) = |k|^2$. BUT FROM THE FORMULA OF $E_{\theta_0,\theta}(f)$ ABOVE WE SEE THAT FOR ANY CHOICE OF θ, θ_0 $E_{\theta,\theta_0}(e^{it\omega}f) = E_{\theta,\theta_0}(f)$ AND THUS ANY CHOICE OF θ, θ_0 CORRESPOND STATIONARY STATE.

NOTE: EQUILIBRIUM STATES ARE CHARACTERIZED BY A SO-CALLED KMS-CONDITION $\Rightarrow$ IN OUR CASE THE EQUILIBRIUM STATE IS UNIQUE.

4

a) It's enough to proof the results at algebraich level.

- We shall use the following basic result:
  - $f: \mathbb{R}^2 \to \mathbb{R}$, s.t. $\frac{\partial}{\partial t} f(t,s)$ exists $\forall t$
  - $\alpha, \beta: \mathbb{R} \to \mathbb{R}$ differentiable and $\alpha(t) \le \beta(t)$ $\forall t \in \mathbb{R}$

Then function:
$$F(t) := \int_{\alpha(t)}^{\beta(t)} f(t,s)\, ds$$
has derivative:
$$F'(t) = \beta'(t) f(t, \beta(t)) - \alpha'(t) f(t, \alpha(t)) + \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t}(t,s)\, ds \quad (*)$$

- We can now proof the results: set
$$\tilde{\rho}_t := e^{t\mathcal{L}_0}\rho_0 + \int_0^t e^{(t-s)\mathcal{L}} \mathcal{L}_1 e^{s\mathcal{L}_0} \rho_0\, ds$$

We must show that $\forall \rho_0$ holds:

$$\dot{\tilde{\rho}}_t \equiv \frac{\partial}{\partial t}\tilde{\rho}_t = \mathcal{L}\tilde{\rho}_t$$

- Direct calculation gives:
$$\dot{\tilde{\rho}}_t \overset{(a)}{=} \mathcal{L}_0 (e^{t\mathcal{L}_0}\rho_0) + 1 \cdot \overset{\overset{0}{\parallel}}{e^{(t-t)\mathcal{L}}} \mathcal{L}_1 e^{t\mathcal{L}_0}\rho_0 - 0 \cdot e^{(t-0)\mathcal{L}} \mathcal{L}_1 e^{0\cdot\mathcal{L}_0}\rho_0 + \int_0^t \mathcal{L}\left\{e^{(t-s)\mathcal{L}} \mathcal{L}_1 e^{s\mathcal{L}_0}\rho_0\right\} ds$$

$$\overset{(b)}{=} \mathcal{L}_0 (e^{t\mathcal{L}_0}\rho_0) + \mathcal{L}_1 (e^{t\mathcal{L}_0}\rho_0) + \mathcal{L}\int_0^t e^{(t-s)\mathcal{L}} \mathcal{L}_1 e^{s\mathcal{L}_0}\rho_0\, ds$$

$$\overset{(c)}{=} \mathcal{L}\tilde{\rho}_t$$

- Where in (a) we have used the def. $\frac{d}{dt}(e^{t\mathcal{L}}\rho) = \frac{d}{dt}\left(\sum_{m\ge0} \frac{t^m}{m!}\mathcal{L}^m\rho\right) = \mathcal{L}\left(\sum_{m\ge0}\frac{t^m}{m!}\mathcal{L}^m\rho\right) = \mathcal{L}(e^{t\mathcal{L}}\rho)$

and the formula $(*)$ with the choices:

$\alpha(t) = 1$, $\beta(t) = t$,

$$f_{x,y}(t,s) = \left(e^{(t-s)\mathcal{L}}\mathcal{L}_1 e^{s\mathcal{L}_0}\rho_0\right)(x,y)$$

for any x,y "coordinates".

- NOTE: x,y were introduced to convince the reader that the operator valuedness of $f(t,s)$ is not a problem, at least at the algebraic level.

- In (b) we assumed that $\mathcal{L}$ commutes with the integration. ↑ just a limit of sums...

- In (c) definition of $\tilde{\rho}_t$ was used.

4a) continues...

▷ The second representation is proven similarly:

$$\tilde{\rho}_t := e^{t\mathcal{L}_0}\rho_0 + \int_0^t e^{(t-s)\mathcal{L}_0}\mathcal{L}_2 e^{s\mathcal{L}}\rho_0 \, ds$$

$$\Rightarrow \dot{\tilde{\rho}}_t = \mathcal{L}_0\left(e^{t\mathcal{L}_0}\rho_0\right) + 1\cdot\mathcal{L}_1 e^{t\mathcal{L}}\rho_0 - 0\cdot(\dots)$$
$$+ \int_0^t \mathcal{L}_0\left\{e^{(t-s)\mathcal{L}_0}\mathcal{L}_2 e^{s\mathcal{L}}\rho_0\right\} ds$$

$$= \mathcal{L}\tilde{\rho}_t \; \boxtimes$$

(b) – In order to avoid technicalities we assume that the underlying H-space is finite dimensional: $d := \dim(\mathcal{H}) < \infty$.

– So suppose L_α are given.

– The easiest way to make them traceless is to set

$$\tilde{L}_\alpha := L_\alpha - \frac{1}{d}\mathrm{Tr}(L_\alpha)\cdot\mathbb{1}$$
$$\equiv L_\alpha + c_\alpha \quad , \quad c_\alpha := \frac{1}{d}\mathrm{Tr}(L_\alpha)\mathbb{1} \in \mathbb{C}$$

NOTE: "$\frac{1}{d}\mathbb{1}$" does not make sense when $d = \infty$!

– Let's define $\mathcal{L}_\alpha, \tilde{\mathcal{L}}_\alpha$ by setting:

$$\mathcal{L}_\alpha\rho := 2L_\alpha\rho L_\alpha^\dagger - L_\alpha^\dagger L_\alpha\rho - \rho L_\alpha^\dagger L_\alpha$$
$$\tilde{\mathcal{L}}_\alpha\rho := 2\tilde{L}_\alpha\rho\tilde{L}_\alpha^\dagger - \tilde{L}_\alpha^\dagger\tilde{L}_\alpha\rho - \rho\tilde{L}_\alpha^\dagger\tilde{L}_\alpha$$

– We'll show that:

$$\tilde{\mathcal{L}}_\alpha\rho = \mathcal{L}_\alpha\rho + i\left[K_\alpha, \rho\right]$$

where $K_\alpha^\dagger = K_\alpha$ and thus

$$\mathcal{L}\rho = -i\left[H, \rho\right] + \sum_\alpha \mathcal{L}_\alpha\rho$$
$$= -i\left[\tilde{H}, \rho\right] + \sum_\alpha \tilde{\mathcal{L}}_\alpha\rho$$

if one sets:

$$\tilde{H} = H + \sum_\alpha K_\alpha.$$

– To see what K_α should be we compute: (drop α's: $\tilde{\mathcal{L}} \equiv \tilde{\mathcal{L}}_\alpha$, $\mathcal{L} \equiv \mathcal{L}_\alpha$ [not original $\mathcal{L}$!])

$$\tilde{\mathcal{L}}\rho = 2\tilde{L}\rho\tilde{L}^\dagger - \tilde{L}^\dagger\tilde{L}\rho - \rho\tilde{L}^\dagger\tilde{L}$$
$$= 2(L+c)\rho(L^\dagger+\bar{c}) - (L^\dagger+\bar{c})(L+c)\rho - \rho(L^\dagger+\bar{c})(L+c)$$
$$= \boxed{2L\rho L^\dagger} + 2(\bar{c}L)\rho + 2\rho(cL^\dagger) \boxed{+2|c|^2\rho} \leftarrow \text{cancels!}$$
$$\boxed{- L^\dagger L\rho} - (\bar{c}L)\rho - (cL^\dagger)\rho \boxed{- |c|^2\rho}$$
$$\boxed{- \rho L^\dagger L} - \rho(\bar{c}L) - \rho(cL^\dagger) \boxed{- |c|^2\rho}$$
$$= \mathcal{L}\rho + (\bar{c}L - cL^\dagger)\rho - \rho(\bar{c}L - cL^\dagger)$$
$$= \mathcal{L}\rho + i\Big[\underbrace{\tfrac{1}{i}(\bar{c}L - cL^\dagger)}_{\equiv K}, \rho\Big]$$

– So: $K_\alpha = \frac{1}{i}(\bar{c}_\alpha L_\alpha - c_\alpha L_\alpha^\dagger)$

(5) - We have in coordinate representation:

$$H\psi(x) = \frac{-1}{2m}\Delta\psi(x)$$
$$= \frac{1}{2m}\sum_{y:|x-y|=1}(\psi(x)-\psi(y)) \quad , \quad x \in \mathbb{Z}^d$$

- Since evolution of the particle purely unitary we have for $\rho(0) = |\psi_0\rangle\langle\psi_0| \Rightarrow \rho(t) = |\psi(t)\rangle\langle\psi(t)|$

- We want to show that for $\psi_0(x) = \delta_0(x)$ one has:

$$\mathrm{Tr}\{X^2\rho(t)\} \overset{(*)}{\sim} t^2 \quad , \quad X^2 \equiv \sum_j x_j^2$$

- Since Hamiltonian diagonalizes in F-space we will work in Fourier-space

- The position operator: $(X_j\psi)(x) = x_j\psi(x)$ then takes the form:

$$(X_j\psi)^\wedge(p) = \sum_x e^{-ip\cdot x} x_j\psi(x)$$
$$= \left(i\frac{\partial}{\partial p_j}\right)\left\{\sum_x e^{-ip\cdot x}\psi(x)\right\} = i\frac{\partial}{\partial p_j}\hat{\psi}(p)$$

- This gives $X^2 = -\sum_j \frac{\partial^2}{\partial p_j^2} \equiv -\Delta_p$

- Since $\psi_0 = \delta_0$ ($\Rightarrow \hat{\psi}_0 = 1$) and $\underbrace{(e^{-itH}\psi_0)^\wedge(p)}_{\hat{\psi}_t(p)} = e^{-it\mathcal{E}(p)}\hat{\psi}_0(p)$

with $\mathcal{E}(p) = \frac{1}{2m}(-\Delta)^\wedge(p)$
$$= \frac{2}{m}\sum_j \sin^2\left(\frac{p_j}{2}\right)$$

We get:

$$\mathrm{Tr}\{X^2\rho(t)\} = \langle\psi_t, X^2\psi_t\rangle = \int_{[0,2\pi]^d}\overline{\hat{\psi}_t(p)}\left\{-\sum_j\frac{\partial^2}{\partial p_j^2}\right\}\hat{\psi}_t(p)\,dp$$

$$= -\sum_j\int e^{+it\mathcal{E}(p)}\left\{\left(-it\frac{\partial\mathcal{E}}{\partial p_j}(p)\right)^2 + \left(-it\frac{\partial^2\mathcal{E}}{\partial p_j^2}(p)\right)\right\}e^{-it\mathcal{E}(p)}\,dp$$

$$= \underbrace{\left(\sum_j\int_{[0,2\pi]^d}\left(\frac{\partial\mathcal{E}}{\partial p_j}(p)\right)^2 dp\right)}_{\in(0,\infty)}\cdot t^2 + i\underbrace{\left(\sum_j\int_{[0,2\pi]^d}\frac{\partial^2\mathcal{E}}{\partial p_j^2}(p)\,dp\right)t}_{=0}$$

$=0$ since integrant is real while x^2 is self-adjoint.

- Because $\int \sin^2\left(\frac{p}{2}\right)dp \in (0,\infty)$

(*) There was a typo in the exam: used to be $\sim t$. (This is of course diffusive).

(6) WE HAVE: $H = H_0 + H_I$ WITH:

$$H_0 = \varepsilon \sum_j Z_j \otimes 1_E + 1_S \otimes \sum_\ell \omega_\ell a_\ell^\dagger a_\ell$$

$$H_I = \sum_{j\ell}' Z_j \otimes \left(g_{j\ell} a_\ell^\dagger + \bar{g}_{j\ell} a_\ell \right)$$

(a) Let's calculate $\tilde{H}_I(t) = e^{+itH_0} H_I e^{-itH_0}$:

$$\tilde{H}_I(t) = e^{itH_S} \otimes e^{itH_E} \left(\sum_{j\ell}' Z_j \otimes \left(g_{j\ell} a_\ell^\dagger + \bar{g}_{j\ell} a_\ell \right) \right) e^{-itH_S} \otimes e^{-itH_E}$$

$$= \sum_{j\ell} Z_j \otimes \left(g_{j\ell} \left(e^{itH_E} a_\ell^\dagger e^{-itH_E} \right) - \bar{g}_{j\ell} \left(e^{itH_E} a_\ell e^{-itH_E} \right) \right), \quad \text{SINCE } [Z_j, H_S] = 0$$

Now, $e^{itH_E} a_\ell^\dagger e^{-itH_E} = e^{i\omega_\ell t} a_\ell^\dagger$, SO ONE GETS:

$$\tilde{H}_I(t) = \sum_{j\ell} Z_j \otimes \left(\tilde{g}_{j\ell}(t) a_\ell^\dagger + \tilde{g}^*_{j\ell}(t) a_\ell \right), \quad \tilde{g}_{j\ell}(t) = e^{+i\omega_\ell t} g_{j\ell}$$

Thus,

$$[\tilde{H}_I(t), \tilde{H}_I(s)] = \sum_{jj'} \sum_{\ell\ell'} \underbrace{\left[Z_j \otimes \left(\tilde{g}_{j\ell} a_\ell^\dagger + \tilde{g}^*_{j\ell} a_\ell \right), Z_{j'} \otimes \left(\tilde{g}_{j'\ell'} a_{\ell'}^\dagger + \tilde{g}^*_{j'\ell'} a_{\ell'} \right) \right]}_{= Z_j Z_{j'} \otimes \underbrace{\left[\tilde{g}_{j\ell} a_\ell^\dagger + \tilde{g}^*_{j\ell} a_\ell, \tilde{g}_{j'\ell'} a_{\ell'}^\dagger + \tilde{g}^*_{j'\ell'} a_{\ell'} \right]}_{= \left(\tilde{g}^*_{j\ell} \tilde{g}_{j'\ell'} \delta_{\ell\ell'} + \tilde{g}_{j\ell} \tilde{g}^*_{j'\ell'} (-\delta_{\ell\ell'}) \right) \cdot 1_E}}$$

$$= \sum_\ell' \left\{ \sum_{jj'} w^\ell_{jj'}(t-s) \, Z_j Z_{j'} \right\} \otimes 1_E,$$

where: $w^\ell_{jj'}(\tau) = -\left(e^{i\omega_\ell \tau} g_{j\ell} g^*_{j'\ell} - e^{-i\omega_\ell \tau} g^*_{j\ell} g_{j'\ell} \right)$

- If $j = j' = 0$ (ONE qubit) set $g_\ell := g_{0\ell}$, AND USE $Z_j^2 = 1$ TO obtain $w^\ell_{00}(\tau) = -2i \cdot |g_\ell|^2 \sin(\omega_\ell \tau)$, so that

$$[\tilde{H}_I(t), \tilde{H}_I(s)] = -2i \sum_\ell' |g_\ell|^2 \sin\left[\omega_\ell (t-s) \right] \cdot 1_{SE} =: -h(t-s) 1_{SE} = h(s-t) 1_{SE}$$

(b) $\frac{d}{dt}(A^M) = \sum_{j=0}^{M-1}' A^{M-j-1} \dot{A} A^j$, where

$$\dot{A} A^j = \left\{ [\dot{A}, A] + A\dot{A} \right\} A^{j-1} = -c(t) A^{j-1} + A \left(\dot{A} A^{j-1} \right) = \cdots = -c(t) \cdot j A^{j-1} + A^j \dot{A}$$

$$= \sum_{j=0}^{M-1}' A^{M-j-1} \left\{ A^j \dot{A} - c(t) j A^{j-1} \right\}$$

$$= M \cdot A^{M-1} \dot{A} - c(t) \left\{ \sum_{j=0}^{M-1}' j \right\} A^{M-2} = M A^{M-1} \dot{A} - \tfrac{1}{2} M(M-1) A^{M-2} c(t)$$

- Actually, one doesn't need to assume that $[A, \dot{A}] = c(t)\mathbb{1}$
- The result holds, with the same proof, provided $C_t := [A_t, \dot{A}_t]$ commutes with $A_t, \dot{A}_t$, $\forall t$: $[C_t, A_t] = [C_t, \dot{A}_t] = 0$

6
$=i\eta$
e.
NEN

- From (D) it follows that when $C_t := [A_t, \dot{A}_t]$ satisfies $[C_t, A_t] = [C_t, \dot{A}_t] = 0$, then

$$\frac{d}{dt}\left(e^{A_t}\right) = \sum_{l \geq 0} \frac{1}{l!} \frac{d}{dt}\left(A_t^l\right) = \sum_{l \geq 0} \frac{1}{l!} \left\{ l \cdot A_t^{l-1} \dot{A}_t - \frac{1}{2} l(l-1) A_t^{l-2} C_t \right\}$$

$$= e^{A_t} \cdot \left(\dot{A}_t - \frac{1}{2} C_t \right) \qquad (1)$$

- This implies that the "average" Hamiltonian $\bar{H}_t := \int_0^t \tilde{H}_I(s)\, ds$ almost generates the solution $\tilde{U}_t$ of $\partial_t \tilde{U}_t = -i \tilde{H}_I(t) \tilde{U}_t$:

NOTE: ERROR IN THE EXAM PAPER: $\tilde{U}_t \neq e^{-i\tilde{H}_I(t)}$

Indeed, by setting

$$A_t := i \bar{H}_t$$

one gets:

$$C_t := [A_t, \dot{A}_t] = \left[i \int_0^t \tilde{H}_I(s)\, ds, \, i \tilde{H}_I(t) \right] = \int_0^t [\tilde{H}_I(t), \tilde{H}_I(s)]\, ds$$

$$\overset{(b)}{=} \underbrace{\left\{ \sum_{j,j'} \left(\sum_l \int_0^t w^l_{jj'}(t-s)\, ds \right) \cdot Z_j Z_{j'} \right\}}_{=: C_t^S} \otimes \mathbb{1}_E$$

- Because C_t^S is diagonal w.r.t. $Z_1 \otimes \cdots \otimes Z_j$ one has:

$$[C_t, H_0(s)] = [C_t, \tilde{H}_I(s)] = 0, \quad \forall s, t$$

Hence $[C_t, A_t] = [C_t, \dot{A}_t] = 0$, and (1) applies.

- We solve $\tilde{U}_t$ now by decomposing:

$$\tilde{U}_t = e^{i\bar{H}_t} V_t$$

where V_t solves:

$$\dot{V}_t = \frac{d}{dt}\left(e^{i\bar{H}_t} \tilde{U}_t \right) = \left(\frac{d}{dt} e^{i\bar{H}_t} \right) \tilde{U}_t + e^{i\bar{H}_t}\left(-i \tilde{H}_I(t) \tilde{U}_t \right)$$

$(A_t = i\bar{H}_t)$: $$= e^{i\bar{H}_t}\left(i\tilde{H}_I(t) - \frac{1}{2} C_t \right) \tilde{U}_t + e^{i\bar{H}_t}\left(-i\tilde{H}_I(t) \right) \tilde{U}_t$$

$$= -\frac{1}{2} C_t \left(e^{i\bar{H}_t} \tilde{U}_t \right)$$ - since $[C_t, \tilde{H}_I(s)] = 0 \Rightarrow [C_t, \bar{H}_t] = 0$

$$= -\frac{1}{2} C_t V_t$$

- Now, $V_0 = \mathbb{1}$, so $$V_t = \exp\left[-\frac{1}{2} \int_0^t C_s\, ds \right] = \exp\left[-\frac{1}{2} \int_0^t C_s^S\, ds \right] \otimes \mathbb{1}_E$$

- When $N=1$: $Z_j Z_{j'} = Z^2 = \mathbb{1}_S$ and $C_t^S = -\int_0^t h(t-s)\, ds \cdot \mathbb{1}_S$.

and thus $$V_t = \exp\left[\frac{1}{2} \int_0^t \int_0^{t_2} h(t_1 - t_2)\, dt_2\, dt_2 \right] \cdot \mathbb{1}_{SE}$$

so

$$\tilde{U}_t = \exp\left[i \Big(\underbrace{\frac{1}{2i} \int_0^t \int_0^{t_1} h(t_1 - t_2)\, dt_2\, dt_1}_{= \varphi(t)} \Big) \right] \cdot \exp\left[-i \int_0^t \tilde{H}_I(s)\, ds \right]$$

$\Rightarrow$ use (a) to see $\varphi : \mathbb{R} \to \mathbb{R}$ $\Rightarrow$ global phase

6 c) iii) continues...

$[V_\ell, \bar{H}_\ell] = 0$ since $[C^S_s, Z_j] = 0, \forall j, \forall s$

$\underline{N \geq 2:}$

$$\tilde{U}_t = e^{-i\bar{H}_t} V_t = V_t e^{-i\bar{H}_t}$$

$$= \underbrace{\left(\exp\left[-\frac{1}{2}\int_0^t C^S_s ds\right] \otimes \mathbb{I}_E\right)}_{= \tilde{V}(t) \text{ IN THE EXAM PAPER.}} \exp\left[-i\int_0^t \tilde{H}_I(s) ds\right]$$

- Besides being diagonal in Z-coordinates, $\underline{V_t \text{ is also unitary.}}$
- By direct computation:

$$C^S_t = \sum_{jj'}' \left\{ \sum_\ell' \int_0^t w^\ell_{jj'}(t-s) ds \right\} Z_j Z_{j'}$$

where

$$w^\ell_{jj'}(\tau) = -e^{i\omega_\ell \tau} \eta_{j\ell} \bar{\eta}_{j'\ell} + e^{-i\omega_\ell \tau} \bar{\eta}_{j\ell} \eta_{j'\ell}$$

$$= -2i \cdot \mathrm{Im}\left[e^{i\omega_\ell \tau} \eta_{j\ell} \bar{\eta}_{j'\ell}\right]$$

- All in all, in the eigenbasis of $Z_0 \otimes Z_1 \otimes \dots \otimes Z_{N-2}$ $\tilde{V}(t) \equiv V_t$ is a diagonal matrix with non zero elements of the form $e^{i\theta}, \theta \in \mathbb{R}$.

6d) Let's start by computing

$$\bar{H}(t) := \int_0^t \tilde{H}_I(s)\,ds$$
$$\overset{(N=1)}{=} Z \otimes \sum_\ell \left\{ \left(\int_0^t e^{i\omega_\ell s} ds\right) \eta_\ell a_\ell^\dagger + \left(\int_0^t e^{-i\omega_\ell s} ds\right) \bar{\eta}_\ell \cdot a_\ell \right\}$$
$$= i \cdot Z \otimes \sum_\ell \left\{ \frac{\alpha_\ell(t)}{2} a_\ell^\dagger - \frac{\bar{\alpha}_\ell(t)}{2} a_\ell \right\}$$

with $\alpha_\ell(t) = \frac{2}{i}\left(\int_0^t e^{i\omega_\ell s} ds\right) \cdot \eta_\ell$

$$= 2\eta_\ell \cdot \frac{1 - e^{i\omega_\ell t}}{\omega_\ell}$$

NOTE: ERROR IN THE EXAM PAPER:

$D_\ell(z) = e^{z a_\ell^\dagger - \bar{z} a_\ell}$

NOT: $D_\ell(z) = e^{z a_\ell^\dagger + \bar{z} a_\ell}$

SO:

$$\tilde{U}_t \equiv e^{i g(t)} e^{-i\bar{H}(t)}$$
$$= e^{i g(t)} \exp\left[Z \otimes \sum_\ell \left(\frac{\alpha_\ell(t)}{2} a_\ell^\dagger - \frac{\bar{\alpha}_\ell(t)}{2} a_\ell \right)\right]$$
$$= e^{i g(t)} \prod_\ell \left\{ |0\rangle\langle 0| \otimes D\left(\frac{\alpha_\ell(t)}{2}\right) + |1\rangle\langle 1| \otimes D\left(-\frac{\alpha_\ell(t)}{2}\right)\right\}$$

where $\begin{cases} Z|0\rangle = +|0\rangle, \quad Z|1\rangle = -|1\rangle, \quad P_q := |q\rangle\langle q|, \ q \in \{0,1\} \\ D_\ell(\alpha) = \exp\left[\alpha a_\ell^\dagger - \bar{\alpha} a_\ell\right] \end{cases}$

- Now, we have:

$$\tilde{\rho}_S(t) = \mathrm{Tr}_E\left\{ \tilde{U}_t (\rho_S \otimes \rho_E) \tilde{U}_t^\dagger \right\}$$
$$= \mathrm{Tr}_E\left\{ \left[\sum_q |q\rangle\langle q| \otimes \prod_\ell D_\ell\left((-1)^q \frac{\alpha_\ell(t)}{2}\right)\right] \rho_S \otimes \rho_E \left[\sum_{q'} |q'\rangle\langle q'| \otimes \prod_{\ell'} D_\ell^\dagger\left((-1)^{q'} \frac{\alpha_{\ell'}(t)}{2}\right)\right]\right\}$$
$$= \sum_{q,q'=0}^{1} \langle q|\rho|q'\rangle \cdot \mathrm{Tr}\left\{ \prod_\ell \left[D_\ell\left(-(-1)^{q'}\frac{\alpha_\ell(t)}{2}\right) D_\ell\left((-1)^q \frac{\alpha_\ell(t)}{2}\right)\right] \cdot \rho_E \right\} \cdot |q\rangle\langle q'|$$

- But $D(\alpha)D(\beta) = e^{+i\,\mathrm{Im}(\alpha^*\beta)} D(\alpha+\beta), \quad \forall \alpha,\beta \in \mathbb{C}$

so $$[\cdots] = \exp\left[\frac{i}{4} \underbrace{\mathrm{Im}\left((-1)^{q+q'+1} |\alpha_\ell(t)|^2\right)}_{=0}\right] \cdot D_\ell\left((1-\delta_{qq'})\alpha_\ell(t)\right)$$

- This can be now written as:

$$\langle q|\tilde{\rho}_S(t)|q'\rangle = \langle q|\rho_S|q'\rangle \cdot \begin{cases} 1 & \text{for } q = q' \\ \mathrm{Tr}\left[\prod_\ell D_\ell(\alpha_\ell(t)) \cdot \rho_E\right] & \text{otherwise.} \end{cases}$$

- <u>Physical intuition</u> about the diagonal suppression factors (1 above):

- Since $[\tilde{H}_I, Z_j] = 0$ spin is constant of motion.

6e) If $\rho_E = |\Omega\rangle\langle\Omega|$ we have

$$x_{mm}(t) = \langle \Omega, \prod_\ell D_\ell(\alpha_\ell(t))\, \Omega \rangle$$
$$= \prod_\ell \exp\left[-\frac{|\alpha_\ell(t)|^2}{2}\right]$$
$$= \exp\left[-\sum_\ell{}' 4|g_\ell|^2 \frac{1-\cos(\omega_\ell t)}{\omega_\ell^2}\right]$$

- For times t which are short compared to the field dynamics, e.g. $t \ll \frac{1}{\omega_\ell}$, $\frac{1-\cos(\omega_\ell t)}{\omega_\ell^2} = \frac{t^2}{2} + O(\omega_\ell^2 t^4)$

we thus obtain <u>Gaussian decay.</u>

f) Thermal state for $\rho_E := \rho_\beta \equiv \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$ where $H = d\Gamma(h)$

$$x_{mM}(t) = \mathrm{Tr}\left\{\prod_\ell D_\ell(\alpha_\ell)\, \rho_\beta\right\}, \quad \alpha_\ell \equiv \alpha_\ell(t), \quad m \neq M$$

- This can be now computed easily by writing it in the form $x_{mM}(t) = \mathrm{Tr}\left\{e^{i\Phi(f)}\rho_\beta\right\}$, $\Phi(f) = a^\dagger(f) + a(f)$

and then using the result of exercise (9.4) to evaluate RHS.

- To find out f in terms of α_ℓ we compute:

$$\mathrm{Tr}\left\{\prod_\ell D_\ell(\alpha_\ell)\rho_\beta\right\} = \mathrm{Tr}\left\{\exp\left[\sum_\ell (\alpha_\ell a_\ell^\dagger - \bar\alpha_\ell a_\ell)\right]\rho_\beta\right\}$$
$$= \mathrm{Tr}\left\{\exp\left[i\sum_\ell \left(\left(\frac{\alpha_\ell}{i}\right)a^\dagger(\phi_\ell) + \left(\overline{\frac{\alpha_\ell}{i}}\right)a(\phi_\ell)\right)\right]\rho_\beta\right\}$$
$$= \mathrm{Tr}\left\{e^{i\Phi(f)}\rho_\beta\right\}$$

with $a_\ell \equiv a(\phi_\ell)$, $f = \sum_\ell (-i\alpha_\ell)\phi_\ell$, $h = \sum_\ell{}' \omega_\ell |\phi_\ell\rangle\langle\phi_\ell|$
$\langle\phi_\ell|\phi_{\ell'}\rangle = \delta_{\ell\ell'}$

- Now, exercise (9.4) gives:

$$x_{mM}(t) = \exp\left[-\frac{1}{2}\left\langle f, \frac{1+e^{-\beta h}}{1-e^{-\beta h}} f\right\rangle\right]$$

The exponent equals:

$$-\frac{1}{2}\left\langle f, \frac{1+e^{-\beta h}}{1-e^{-\beta h}} f\right\rangle = -\sum_\ell \frac{|-i\alpha_\ell|^2}{2}\left\langle \phi_\ell, \frac{1+e^{-\beta\omega_\ell}}{1-e^{-\beta\omega_\ell}}\phi_\ell\right\rangle$$
$$= -\sum_\ell{}' \frac{|\alpha_\ell(t)|^2}{2}\coth\left(\frac{\beta\omega_\ell}{2}\right)$$

- <u>Note:</u> of course we could have computed $x_{mM}(t)$ by using $D(\alpha) = e^{\alpha a^\dagger - \bar\alpha a} = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\bar\alpha a}$, and commuted $a^\dagger$'s to the left of all a's, etc.

6h) Generally from part g): just like when N=1:

$$\overline{H}_t = \int_0^t \tilde{H}_I(s)\, ds$$

$$= i \sum_j Z_j \otimes \sum_l \underbrace{\left\{ \left(\frac{\alpha_{jl}(t)}{2}\right) a_l^\dagger - \overline{\left(\frac{\alpha_{jl}(t)}{2}\right)} a_l \right\}}_{=:\ \Phi_l\left(\frac{\alpha_{jl}(t)}{2}\right)} \equiv i \sum_j{}' Z_j \otimes \sum_l \Phi_l\left(\frac{\alpha_{jl}(t)}{2}\right)$$

where $\alpha_{jl}(t) = \frac{2}{i}\left(\int_0^t e^{i\omega s} ds\right) \cdot g_{jl}$

$$= 2 g_{jl} \cdot \frac{1 - e^{i\omega_l t}}{\omega_l}$$

Thus evolution is given by:

$$\tilde{U}_t = (\tilde{V}_t \otimes 1_E) \exp\left[-i \overline{H}_t\right]$$

$$= (\tilde{V}_t \otimes 1_E) \exp\left[\sum_j{}' Z_j \otimes \sum_l \Phi_l\left(\frac{\alpha_{jl}(t)}{2}\right)\right]$$

- Let N=2, and let only $l=1$ couplings g_{j1} be non-zero:

$$\tilde{U}_t |m_0, m_1\rangle \otimes \Omega = (\tilde{V}_t \otimes 1_E) |m_0, m_1\rangle \otimes \left| (-1)^{m_0} \frac{\alpha_{01}}{2} + (-1)^{m_1} \frac{\alpha_{11}}{2} \right\rangle$$

- Moreover, $\tilde{V}_t |m_0 m_1\rangle = e^{i\theta_{m_0 m_1}(t)} |m_0, m_1\rangle$ for some $\theta_{m_0 m_1}(t) \in \mathbb{R}$

- Thus:

$$\tilde{U}_t(\phi \otimes \Omega) = e^{i\theta_{00}(t)} c_{00} |00\rangle \otimes \left| \frac{\alpha_{01}(t) + \alpha_{11}(t)}{2} \right\rangle$$

$$+ e^{i\theta_{11}(t)} c_{11} |11\rangle \otimes \left| \frac{-\alpha_{01}(t) - \alpha_{11}(t)}{2} \right\rangle$$

$$\tilde{U}_t(\psi \otimes \Omega) = e^{i\theta_{10}(t)} c_{10} |10\rangle \otimes \left| \frac{-\alpha_{01}(t) + \alpha_{11}(t)}{2} \right\rangle$$

$$+ e^{i\theta_{01}(t)} c_{01} |01\rangle \otimes \left| \frac{\alpha_{01}(t) - \alpha_{11}(t)}{2} \right\rangle$$

6i) Let's generalize the result of h):

- Let $P_{\underline{m}} := |\underline{m}\rangle\langle\underline{m}|$, $|\underline{m}\rangle \equiv |m_0, \dots, m_{N-1}\rangle$, $\underline{m} \equiv (m_0, \dots, m_{N-1}) \in \{0,1\}^N$ denote the projector s.t. $Z_0 \otimes Z_1 \otimes \dots \otimes Z_{N-2} = \sum_{\underline{m}}' (-1)^{\sum m_j} P_{\underline{m}}$
- Then

$$
\begin{aligned}
e^{-i\tilde{H}_I(t)} &= V_t \cdot \exp\left[\sum_j Z_j \otimes \sum_\ell' \Phi_\ell\left(\frac{\alpha_{j\ell}}{2}\right)\right] \\
&= V_t \cdot \exp\left[\sum_{\underline{m}} \sum_j' (-1)^{m_j} P_{\underline{m}} \otimes \sum_\ell' \Phi_\ell(\dots)\right] \\
&= V_t \cdot \exp\left[\sum_{\underline{m}}' P_{\underline{m}} \otimes \sum_\ell' \Phi_\ell\left(\sum_j' (-1)^{m_j} \frac{\alpha_{j\ell}}{2}\right)\right] \\
&= V_t \cdot \sum_{\underline{m}}' P_{\underline{m}} \otimes e^{\sum_\ell' \Phi_\ell\left(\sum_j' (\dots)\right)} \\
&= V_t \cdot \sum_{\underline{m}} P_{\underline{m}} \otimes \prod_\ell D_\ell\Big(\overbrace{\sum_j' (-1)^{m_j} \frac{\alpha_{j\ell}(t)}{2}}^{=:\alpha_\ell(\underline{m})}\Big)
\end{aligned}
$$

Even though not strictly necessary let's compute $\tilde{\rho}_S(t)$:

$$
\begin{aligned}
\tilde{\rho}_S(t) &\equiv \mathrm{Tr}_E\{\tilde{U}_t\, \rho_S \otimes \rho_E\, \tilde{U}_t^\dagger\} \\
&= \sum_{\underline{m},\underline{m}'} \mathrm{Tr}_E\left\{(V_t P_{\underline{m}})\, \rho_S\, (P_{\underline{m}'} V_t^\dagger) \otimes \left(\prod_\ell D_\ell(\alpha_\ell(\underline{m}))\right) \rho_E \left(\prod_{\ell'} D_{\ell'}^\dagger(\alpha_{\ell'}(\underline{m}'))\right)\right\} \\
&= \sum_{\underline{m},\underline{m}'} (V_t P_{\underline{m}})\, \rho_S\, (P_{\underline{m}'} V_t^\dagger) \cdot \mathrm{Tr}\left\{\prod_\ell \left[D_\ell(-\alpha_\ell(\underline{m}'))\, D_\ell(\alpha_\ell(\underline{m}))\right] \rho_E\right\} \quad (A)
\end{aligned}
$$

where we have used $D_\ell^\dagger(\alpha) = D_\ell(-\alpha)$. Now, $D_\ell(-\alpha')D_\ell(\alpha) = e^{i\,\mathrm{Im}(\bar{\alpha}\cdot\alpha')} D(\alpha - \alpha')$

- Thus: $\mathrm{Tr}\left\{\prod_\ell [\dots] \rho_E\right\}$

$$
= e^{i\,\mathrm{Im}\left(\sum_\ell \overline{\alpha_\ell(\underline{m})}\, \alpha_\ell(\underline{m}')\right)} \cdot \mathrm{Tr}\left\{\prod_\ell D_\ell(\alpha_\ell(\underline{m}) - \alpha_\ell(\underline{m}'))\, \rho_E\right\} \quad (B)
$$

- The phase factor is not particularly interesting, so we simply define $\Phi_I(\underline{m},\underline{m}') := \mathrm{Im}\left[\sum_\ell \overline{\alpha_\ell(\underline{m})}\, \alpha_\ell(\underline{m}')\right]$
- The factor inside $\mathrm{Tr}\{\cdot\}$ is more interesting. Let's rewrite it as:

$$
\begin{aligned}
\alpha_\ell(\underline{m}) - \alpha_\ell(\underline{m}') &= \frac{1}{2}\sum_j' \left[(-1)^{m_j} - (-1)^{m_j'}\right] \alpha_{j\ell} \\
&= \sum_j' \alpha_{j\ell} \cdot \frac{1}{2}\left[(1-2m_j) - (1-2m_j')\right] \\
&= \sum_j' \alpha_{j\ell} \cdot (m_j' - m_j)
\end{aligned}
$$

- <u>General suppression factor</u> is thus given by (combine (A) + (B))

$$
\underline{\underline{x_{\underline{m},\underline{M}}(t)}} = e^{i\Phi_V(\underline{m},\underline{M})}\, e^{i\Phi_I(\underline{m},\underline{M})} \cdot \mathrm{Tr}\left\{\prod_\ell D_\ell\left(\sum_{j\ell} \alpha_{j\ell}(t) \cdot (m_j' - m_j)\right) \rho_E\right\}
$$

where $\Phi_V(\underline{m},\underline{M})$ comes from V_t and $V_t^\dagger$ - recall, V_t is diagonal in $Z_0 \otimes \dots \otimes Z_{N-2}$ - eigenbasis. (6j)

5j) - In the case of $\eta_{jl} = \eta_l$ we have also

$$\alpha_{jl}(t) = 2\eta_{jl} \cdot \frac{1 - e^{i\omega_l t}}{\omega_l} = \alpha_j(t).$$

- Now, we are asked to find subspaces $\mathcal{H}_S^{(Q)} \subset \mathcal{H}_S$ s.t.

$$\langle \psi_1, \tilde{\rho}_S(t)\psi_2\rangle = \langle \psi_1, \underbrace{\tilde{\rho}_S(0)}_{= \rho_S}\psi_2\rangle, \forall t \geq 0$$

provided $\psi_1, \psi_2 \in \mathcal{H}_S^{(Q)}$.

- We recall from (i) that

$$x_{\underline{m},\underline{M}}(t) = \frac{\langle \underline{m}|\tilde{\rho}_S(t)|\underline{M}\rangle}{\langle \underline{m}|\rho_S|\underline{M}\rangle} = e^{i\Phi_V(\underline{m},\underline{M})}\, e^{i\Phi_I(\underline{m},\underline{M})} \cdot \mathrm{Tr}\Big\{\overbrace{\prod_l D_l\Big(\sum_j{}'(m_j'-m_j)\cdot\alpha_{jl}(t)\Big)}^{U_l(\underline{m})}\rho_E\Big\}$$

- Let's start with the last factor which describes the decay of the absolute values of the (non-diagonal) terms.

- Since for any ρ_E and any unitary operation U on $\mathcal{H}_E$ one has:

$$\big|\mathrm{Tr}\,(U\rho_E)\big| = \Big|\mathrm{Tr}\Big\{U\sum_j p_j|\varphi_j\rangle\langle\varphi_j|\Big\}\Big| \quad , \; p_j \geq 0, \; \langle\varphi_i|\varphi_j\rangle = \delta_{ij}$$

$$\leq \sum_j p_j \underbrace{|\langle\varphi_j, U\varphi_j\rangle|}_{\leq 1}$$

one sees that $|x_{\underline{m},\underline{M}}(t)| = 1$ iff $D_l\Big(\sum_j{}'(m_j'-m_j)\,\alpha_{jl}(t)\Big) = \mathbb{1}_E, \forall l$

- This is clearly the case iff

$$\sum_j{}'(m_j'-m_j)\cdot\alpha_{jl}(t) \overset{\alpha_{jl}=\alpha_l}{=} \alpha_l\cdot\sum_j{}'(m_j'-m_j) = 0$$

i.e. $\sum_j{}' m_j = \sum_j{}' m_j'$

- Let $\mathcal{H}_S^{(Q)} := \mathrm{span}\Big\{|\underline{m}\rangle : \sum_{j=0}^{N-1}{}' m_j = Q\Big\}, \; Q = 0,1,2,\ldots N-1$

$$\Rightarrow \dim(\mathcal{H}_S^{(Q)}) = \binom{N}{Q}$$

- We must still show that the angles $\Phi_V(\underline{m},\underline{M})$, $\Phi_I(\underline{m},\underline{M})$ remain constant inside each $\mathcal{H}_S^{(Q)}$:

▷ $\Phi_V(\underline{m},\underline{M})$: from g) we see that

$$V_t = \exp\Big[-\frac{i}{2}\int_0^t C_s^S\, ds\Big], \quad C_s^S = \Big\{\sum_l\int_0^t w^l(t-s)\,ds\Big\}\Big(\sum_i Z_i\Big)^2$$

- Clearly, $(\sum_i{}' Z_i)$ remains constant in $\mathcal{H}_S^{(Q)}$, so also $\Phi_V(\underline{m},\underline{M}) = 0$ for $|\underline{m}\rangle, |\underline{M}\rangle \in \mathcal{H}_S^{(Q)}$

▷ $\Phi_I(\underline{m},\underline{M}) = \mathrm{Im}\Big[\sum_l\Big(\sum_j(1-2m_j)\frac{\overline{\alpha_l}}{2}\Big)\Big(\sum_{j'}(1-2m_{j'}')\frac{\alpha_l}{2}\Big)\Big]$

$= 0$

$\Rightarrow$ so $x_{\underline{m},\underline{M}}(t) = 1$ when $|\underline{m}\rangle, |\underline{M}\rangle \in \mathcal{H}_S^{(Q)} \Rightarrow$ Each $\mathcal{H}_S^{(Q)}$ is decoherence free subspace of dim $\binom{N}{Q}$

6j) CONTINUES...

- MAXIMUM DECAY RATE IS ACHIEVED WHEN

$\varphi_1 \in \mathcal{H}_S^{(0)}$ AND $\varphi_2 \in \mathcal{H}_S^{(N)}$

- Let's TAKE $\rho_E = |z_1, z_2, \dots\rangle\langle z_1, z_2 \dots|$, coherent state $a_\ell |z_1, z_2, \dots\rangle = z_\ell |z_1, \dots\rangle$:

$$\left| \mathrm{Tr}\left\{ \prod_\ell D_\ell \Big(\sum_j \alpha_{j\ell} \cdot \underbrace{(m_j' - m_j)}_{=1 \ \forall j} \Big) |z\rangle\langle z| \right\} \right|$$

$$= \left| \left\langle \Big\{ z_\ell + \sum_j \alpha_{j\ell} \Big\} \Big| \{ z_\ell \} \right\rangle \right| = \prod_\ell \left| \left\langle z_\ell + \sum_j \alpha_{j\ell} \Big| z_\ell \right\rangle \right|$$

$$= \exp\left[-\frac{1}{2} \sum_\ell \Big| \sum_j \alpha_{j\ell}(t) \Big|^2 \right]$$

- SO WHEN $\alpha_{j\ell}(t) = \alpha_\ell(t)$ ONE OBTAINS

$$= \exp\left[-\frac{N^2}{2} \sum_\ell |\alpha_\ell(t)|^2 \right]$$

(FOR SMALL TIMES)

- LIKE IN 6e) THIS IS GAUSSIAN DECAY, but with an amplitude increased by N^2!

6k) Let $\eta_{j\ell} = \delta_{j\ell}$, AND $\rho_\beta = \dfrac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$. THEN (from 6i):

$$x_{\underline{m}\,\underline{m}}(t) = e^{i(\Phi_V + \Phi_I)(\underline{m}, \underline{m})} \mathrm{Tr}\left\{ \prod_\ell D_\ell \left(\alpha_\ell \cdot (m_\ell' - m_\ell) \right) \rho_\beta \right\}$$

$$= \text{---}''\text{---} \cdot \mathrm{Tr}\left\{ \Big[\bigotimes_\ell D_\ell(\alpha_\ell \cdot (m_\ell' - m_\ell)) \Big] \Big[\bigotimes_\ell \rho_\beta^{(\ell)} \Big] \right\}$$

$$= \text{---}''\text{---} \ \mathrm{Tr}\left\{ \bigotimes_\ell \Big[D_\ell(\dots) \rho_\beta^{(\ell)} \Big] \right\}$$

$$= \text{---}''\text{---} \ \prod_\ell \mathrm{Tr}\left\{ D_\ell \big(\alpha_\ell(t) \cdot (m_\ell' - m_\ell) \big) \rho_\beta^{(\ell)} \right\}$$

$$\overset{(6f)}{=} \text{---}''\text{---} \ \exp\left[-\sum_\ell (m_\ell' - m_\ell)^2 \frac{|\alpha_\ell(t)|^2}{2} \coth\left(\frac{\beta \omega_\ell}{2} \right) \right]$$

- Now, if $\omega_\ell = \omega$ as well THEN WE obtain:

$$|x_{\underline{m}\,\underline{m}}(t)| = e^{-\| \underline{m} - \underline{m} \|_H \frac{|\alpha(t)|^2}{2} \coth\left(\frac{\beta \omega}{2} \right)}$$

- THIS shows that the decay is $-\| \underline{m} - \underline{m} \|_H$ TIMES faster as in the single particle case. (6f)

- Moreover, there are no decoherence free subspaces.

(7) ▷ Let $\vartheta_j : [0,1]^m \to [0,1]$, $j=1\dots m$ s.t. $\{\vartheta_1(x),\dots,\vartheta_m(x)\} = \{x_1,\dots,x_m\}$

And $\vartheta_j(x) \le \vartheta_{j+1}(x)$ with $x \equiv (x_1,\dots,x_m) \in [0,1]^m$

▷ THEN: $\forall f \in C([0,1]^m;\mathbb{R})$ holds:

$$A := \mathbb{E}\left(f(V_1^m,V_2^m,\dots,V_m^m)\right) = \int_{[0,1]^m} f(\vartheta_1(x),\dots,\vartheta_m(x))\,dx_1\dots dx_m$$

- Let's associate to any $x\in[0,1]^m$ a permutation $\pi\in S_m$ s.t. $x_{\pi(j)} \le x_{\pi(j+1)}$ $\forall j$, e.g. $\vartheta_j(x) = x_{\pi(j)}$, $\forall j$

THEN

$$A = \sum_{\pi\in S_m} \int_{\Delta_\pi} f(\vartheta_1(x)\dots\vartheta_m(x))\,dx_1\dots dx_m$$

where $\Delta_\pi = \{x\in[0,1]^m : x_{\pi(i)} \le x_{\pi(i+1)}, \forall i = 1,\dots,m-1\}$ and this

$m(\Delta_\pi \cap \Delta_{\pi'}) = 0$, for $\pi \ne \pi'$ and m Lebesgue measure. But:

$$A = \sum_{\pi\in S_m} \int_0^1\int_0^{x_{\pi(m)}}\cdots\int_0^{x_{\pi(3)}}\int_0^{x_{\pi(2)}} f(x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(m-1)},x_{\pi(m)})\,dx_{\pi(1)}\,dx_{\pi(2)}\dots dx_{\pi(m)}$$

$$= m!\cdot\int_0^1\int_0^{x_m}\cdots\int_0^{x_3}\int_0^{x_2} f(x_1,x_2,\dots,x_m)\,dx_1\,dx_2\dots dx_{m-1}\,dx_m = (*)$$

- This defines the distribution $(V_1^m,\dots,V_m^m)$ as f was arbitrary.
- We will now complete the proof by showing that

$$B := \mathbb{E}\left[f\left(\frac{T_1}{T_{m+1}},\frac{T_2}{T_{m+1}},\dots,\frac{T_m}{T_{m+1}}\right)\right]$$

equals A for $\forall f \in C([0,1]^m)$.

Indeed:

$$B = \int_0^\infty\int_{t_1}^\infty\cdots\int_{t_{m-1}}^\infty\int_{t_m}^\infty f\left(\frac{t_1}{t_{m+1}},\frac{t_2}{t_{m+1}},\dots,\frac{t_m}{t_{m+1}}\right)\left(\lambda e^{-\lambda(t_{m+1}-t_m)}dt_{m+1}\right)\left(\lambda e^{-\lambda(t_m-t_{m-1})}dt_m\right)\cdots\left(\lambda e^{-\lambda(t_2-t_1)}dt_2\right)\left(\lambda e^{-\lambda t_1}dt_1\right)$$

$$= \lambda^{m+1}\int_0^\infty\cdots\int_{t_m}^\infty e^{-\lambda t_{m+1}}\cdot f\left(\frac{t_1}{t_{m+1}},\dots,\frac{t_m}{t_{m+1}}\right)dt_{m+1}\dots dt_1$$

$$= \lambda^{m+1}\int_0^\infty e^{-\lambda t_{m+1}}\int_0^{t_{m+1}}\cdots\int_0^{t_3}\int_0^{t_2} f\left(\frac{t_1}{t_{m+1}}\dots\frac{t_m}{t_{m+1}}\right)dt_1\,dt_2\dots dt_m\,dt_{m+1}$$

Now, we make a change-of-variables: $t_j \mapsto \frac{t_j}{t_{m+1}} =: x_j$ for $j=1\dots m$

$$B = \lambda^{m+1}\int_0^\infty t_{m+1}^m e^{-\lambda t_{m+1}}\left[\int_0^1\cdots\int_0^{x_3}\int_0^{x_2} f(x_1,x_2,\dots,x_m)\,dx_1\,dx_2\dots dx_m\right\}dt_{m+1}$$

$$= \underbrace{\left\{\int_0^\infty(\lambda t_{m+1})^m e^{-(\lambda t_{m+1})}\,d(\lambda t_{m+1})\right\}}_{=m!}\cdot\left\{\int_0^1\cdots\int_0^{x_2} f(x_1\dots x_m)\,dx_1\dots dx_m\right\}$$

- This equals $(*)$ and hence A ∎

⑧ $\mathcal{L}_0 = -i[H_{\mathbb{C}}, \cdot]$, $H_{\mathbb{C}} := H - \frac{i}{2}\sum_\alpha \gamma_\alpha L_\alpha^\dagger L_\alpha$

$\mathcal{L}_\alpha := \gamma_\alpha L_\alpha(\cdot)L_\alpha^\dagger$, $\alpha \geq 1$, $\mathcal{S} := \sum_{\alpha\geq 1} \mathcal{L}_\alpha \Rightarrow \mathcal{L} = \mathcal{L}_0 + \mathcal{S}$.

▷ BY USING THE SECOND DUHAMEL EXPANSION OF (4a) REPEATEDLY:

$$
\begin{aligned}
e^{t\mathcal{L}} &= e^{t\mathcal{L}_0} + \int_0^t e^{(t-s_1)\mathcal{L}_0}\mathcal{S}\, e^{s_1\mathcal{L}}\, ds_1 \\
&= e^{t\mathcal{L}_0} + \int_0^t e^{(t-s_1)\mathcal{L}_0}\mathcal{S}\left\{ e^{s_1\mathcal{L}_0} + \int_0^{s_1} e^{(s_1-s_2)\mathcal{L}_0}\mathcal{S}\, e^{s_2\mathcal{L}}\, ds_2 \right\} ds_1 \\
&\;\;\vdots \\
&= e^{t\mathcal{L}_0} + \sum_{M=1}^{N} \int_{t\geq s_1\geq \dots \geq s_M \geq 0} \cdots \int e^{(t-s_1)\mathcal{L}_0} \underset{(1)}{\mathcal{S}}\, e^{(s_1-s_2)\mathcal{L}_0} \underset{(2)}{\mathcal{S}}\, e^{(s_2-s_3)\mathcal{L}_0} \cdots \underset{(M)}{\mathcal{S}}\, e^{(s_M-0)\mathcal{L}_0}\, ds_M \cdots ds_1 \\
&\quad + \int_{t\geq s_1\geq \dots \geq s_{N+1}} e^{(t-s_1)\mathcal{L}_0}\mathcal{S} \cdots \mathcal{S}\, e^{(s_N - s_{N+1})\mathcal{L}}\, ds_{N+1} \cdots ds_1
\end{aligned}
$$

- NOW, AS CUSTOMARY WE ASSUME that $\mathcal{L}_0, \mathcal{L}_\alpha$ are <u>bounded.</u> THEN THE ABOVE SERIES CONVERGES AND WE MAY TAKE THE limit $N \to \infty$. (SEE P. 121 of LECTURE NOTES: $L \mapsto \mathcal{S}$, $t \mapsto (\|\mathcal{L}_0\| t)$)

- BY RENAMING: $t_i = s_{M+1-i}$ for $i = 1, \dots, M$, And USING THE DEF. OF $\mathcal{S}$ ONE ARRIVES TO THE REPRESENTATION OF $e^{t\mathcal{L}}$.

(b) - TO SEE THAT W_t^ω IS <u>NON-LINEAR</u> JUST NOTICE: $W_t^\omega(\alpha\rho) = W_t^\omega(\rho)$, $\forall \alpha \in \mathbb{C}$.

- <u>COMPLETE POSITIVITY:</u> It doesn't MAKE SENSE TO TALK about CP of a NON-LINEAR SUPER OPERATOR AS SUCH THING IS NOT DEFINED (at least ON THIS COURSE) $\Rightarrow$ <u>ERROR</u> IN THE EXAM!

- <u>HOWEVER,</u> UN-NORMALIZED OBJECT, $\tilde{W}_t^\omega$ IS CP. $\forall\omega$:

- FIRST,

$$
\begin{aligned}
(e^{t\mathcal{L}_0}\otimes \mathbb{1})(\rho) &= (e^{t\mathcal{L}_0}\otimes\mathbb{1})\Big(\sum_\alpha \rho_\alpha |\varphi^\alpha\rangle\langle\varphi^\alpha|\Big), \quad \varphi^\alpha = \sum_{i,j} s_i^\alpha \otimes e_i^\alpha, \; \rho_\alpha > 0 \\
&= \sum_\alpha \rho_\alpha \sum_{i,j} (e^{t\mathcal{L}_0}\otimes\mathbb{1})(|s_i^\alpha, e_i^\alpha\rangle\langle s_j^\alpha, e_j^\alpha|) \\
&= \sum_\alpha \rho_\alpha \sum_{i,j} (e^{t\mathcal{L}_0}\otimes\mathbb{1})(|s_i^\alpha\rangle\langle s_j^\alpha| \otimes |e_i^\alpha\rangle\langle e_j^\alpha|) \\
&= \sum_\alpha \rho_\alpha \sum_{i,j} (e^{-itH_{\mathbb{C}}}|s_i^\alpha\rangle\langle s_j^\alpha| e^{itH_{\mathbb{C}}}) \otimes |e_i^\alpha\rangle\langle e_j^\alpha| \\
&= \sum_\alpha \rho_\alpha \Big(\sum_i |\tilde{s}_i^\alpha, e_i^\alpha\rangle\Big)\Big(\sum_j \langle\tilde{s}_j^\alpha, e_j^\alpha|\Big), \quad \tilde{s}_i^\alpha = e^{-itH_{\mathbb{C}}} s_i^\alpha \\
&= \sum_\alpha \rho_\alpha |\tilde{\varphi}^\alpha\rangle\langle\tilde{\varphi}^\alpha| \leftarrow \text{POSITIVE.}
\end{aligned}
$$

- SECOND, $\mathcal{L}_\alpha$ IS CP - EASY TO CHECK! - Similar calculation as above.
- THIRD, Φ_1, Φ_2 are CP $\Rightarrow \Phi_2 \circ \Phi_1$ IS CP
- <u>THUS:</u> $\tilde{W}_t^\omega$ AS A PRODUCT OF CP OPERATORS IS CP ITSELF.

8c) Let's first compute the time derivative:

- Since $e^{t\mathcal{L}}$ is trace-preserving: $\forall \rho \geq 0$:

$$0 = \frac{d}{dt}\mathrm{Tr}(e^{t\mathcal{L}}\rho) = \mathrm{Tr}\{\mathcal{L}e^{t\mathcal{L}}\rho\}$$

so setting $t=0$: $0 = \mathrm{Tr}\{\mathcal{L}\rho\} = 0$. Using $\mathcal{L} = \mathcal{L}_0 + \sum_{\alpha\geq 1}' \mathcal{L}_\alpha$ then gives:

$$\mathrm{Tr}\{\mathcal{L}_0\rho\} = -\sum_{\alpha\geq 1}' \mathrm{Tr}\{\mathcal{L}_\alpha\rho\} \quad \forall \rho \geq 0$$

- This implies:

$$\frac{d}{dt}\mathrm{Tr}\{e^{t\mathcal{L}_0}\rho\} = \mathrm{Tr}\{\mathcal{L}_0(e^{t\mathcal{L}_0}\rho)\}$$

$$= -\sum_{\alpha\geq 1}' \mathrm{Tr}\{\mathcal{L}_\alpha(e^{t\mathcal{L}_0}\rho)\} \quad (*)$$

- Since $e^{t\mathcal{L}_0}$ and $\mathcal{L}_\alpha\rho = \gamma_\alpha L_\alpha\rho L_\alpha^\dagger$, $\gamma_\alpha > 0$ are both positive we see that for $\rho \geq 0$

$$\mathrm{Tr}\{\mathcal{L}_\alpha(e^{t\mathcal{L}_0}\rho)\} \geq 0$$

- Actually, one has here always a strict inequality because $\sum_\alpha L_\alpha^\dagger L_\alpha = \mathbb{1}$ and thus

$$\sum_\alpha' \mathrm{Tr}\{\mathcal{L}_\alpha\rho\} = \sum_\alpha' \mathrm{Tr}\{\overset{\gamma_\alpha > 0}{L_\alpha\rho L_\alpha^\dagger}\} \geq \underbrace{(\min_\alpha \gamma_\alpha)}_{>0}\cdot\mathrm{Tr}\{\underbrace{(\textstyle\sum_\alpha' L_\alpha^\dagger L_\alpha)}_{=\mathbb{1}}\rho\} > 0$$

$\mathrm{Tr}(\rho) = 1 > 0$

- Using $(*)$ then gives:

$$\frac{d}{dt}\mathrm{Tr}\{e^{t\mathcal{L}_0}\rho\} < 0.$$

- <u>Interpretation of $\mathrm{Tr}\{e^{t\mathcal{L}_0}\rho\}$:</u>

  - This is the probability that the system evolves from ρ up to time t without experiencing elementary transformations $\mathcal{L}_\alpha$, $\alpha \geq 1$, due to the environment.
    $\Rightarrow$ This why $H_{\mathbb{C}}$ is not hermitian.

  - Moreover, $\omega \equiv \{(t_1,\alpha_1), (t_2,\alpha_2), \ldots (t_N,\alpha_N), \ldots\} \in \Omega$ specifies a record of interactions with the environment:
    $(t_j,\alpha_j) \cong$ "at time $t = t_j$ there was an interaction $\mathcal{L}_{\alpha_j}$"

8 d) INTERPRETATION:

- Recall: $\Omega := \bigcup_{M\in\mathbb{N}_0} \Omega_M$, $\Omega_M := \{\omega\in\Omega : |\omega| = M\}$ where

  $|\omega| \equiv |\{(t_j,\alpha_j) : 1\leq j\leq M\}| = M$

- Now, by setting

$$M_t^\omega := e^{-i(t-t_M)H_C} L_{\alpha_M} e^{-i(t_M-t_{M-1})H_C} \cdots e^{-i(t_2-t_1)H_C} L_{\alpha_1} e^{-it_1 H_C}$$

one gets

$$\tilde{W}_t^\omega \rho = M_t^\omega \rho (M_t^\omega)^\dagger$$

and consequently the required formula.

- Since $e^{t\mathcal{L}}$ is trace preserving the formula of (a) shows that

$$\int_\Omega (M_t^\omega)^\dagger M_t^\omega \, d\omega = \mathbb{1} \quad \left(\text{here } \int_\Omega (\cdot\cdot)\,d\omega = \sum_{M\in\mathbb{N}_0}{}' \int_{\Omega_M} (\cdot\cdot)\,d\omega \right.$$

$$\text{with } \int_{\Omega_M} (\cdot\cdot)\,d\omega = \sum_{\alpha_1\cdots\alpha_M=2}^{N}{}' \int_0^t \int_0^{t_M} \cdots \int_0^{t_2} (\cdot\cdot)\, dt_1 \cdots dt_{M-1}\, dt_M$$

- This corresponds physically that environment makes a single compound measurement on the path $\omega\in\Omega$ according the measurement operators $\{M_t^\omega : \omega\in\Omega\}$.