10.1

$$D(z) = \exp\left[z a^{\dagger} - \bar{z} a \right]$$

— recall: $z = \sqrt{\frac{\omega}{2}}\, \underset{\in \mathbb{R}}{q} + i\sqrt{\frac{1}{2\omega}}\, \underset{\in \mathbb{R}}{p} \in \mathbb{C}$

$$= \exp\left[\left(\sqrt{\tfrac{\omega}{2}}\, q + i\sqrt{\tfrac{1}{2\omega}}\, p\right) a^{\dagger} - \left(\sqrt{\tfrac{\omega}{2}}\, q - i\sqrt{\tfrac{1}{2\omega}}\, p\right) a \right]$$

$$= \exp\left[i \left\{ p\left(\sqrt{\tfrac{1}{2\omega}}\,(a^{\dagger}+a)\right) - q\left(i\sqrt{\tfrac{\omega}{2}}\,(a^{\dagger}-a)\right) \right\} \right]$$

$$= \exp\left[i\left(p\cdot\hat{Q} - q\cdot\hat{P}\right) \right] \qquad (1)$$

where we have written hat on top of position & momentum operators to avoid mixing them with scalars: $p, q \in \mathbb{R}$

(a) By $e^{tA} B e^{-tA} = B + [A,B]\,t + f([A,[A,B]])$, $f(0)=0$ (2)

and $[\hat{P},\hat{Q}] = \left[i\frac{\partial}{\partial x}, x\right] = i\mathbb{1}$ we get

$$\left[i(p\hat{Q} - q\hat{P}), \hat{Q} \right] = -iq\left[\hat{P},\hat{Q}\right] = q\mathbb{1}$$

$$\left[i(p\hat{Q} - q\hat{P}), \hat{P} \right] = ip\left[\hat{Q},\hat{P}\right] = p\mathbb{1}$$

and thus by (2):

$$D(z)\,\hat{Q}\,D^{-1}(z) = \hat{Q} + q\mathbb{1} \quad \text{and} \quad D(z)\,\hat{P}\,D^{-1}(z) = \hat{P} + p\mathbb{1}$$

(b) Since $|z\rangle = D(z)|\phi_0\rangle$ we have

$$\langle z|\hat{Q}|z\rangle = \langle D(z)\phi_0, \hat{Q}\, D(z)\phi_0\rangle$$
$$= \langle \phi_0, D^{-1}(z)\,\hat{Q}\, D(z)\,\phi_0\rangle$$
$$= \langle \phi_0, D(-z)\,\hat{Q}\, D^{-1}(-z)\,\phi_0\rangle$$
$$= \langle \phi_0, (\hat{Q} + q\mathbb{1})\,\phi_0\rangle = 0 + q\langle\phi_0,\phi_0\rangle = q$$

because: $D^{-1}(z) = D^{\dagger}(z) = \exp\left[i(p\hat{Q} - q\hat{P})\right]^{\dagger}$

$(p\hat{Q} - q\hat{P})$ is self adjoint

$$= \exp\left[(-i)(p\hat{Q} - q\hat{P})\right] = \exp\left[i\left((-p)\hat{Q} - (-q)\hat{P}\right)\right]$$
$$\equiv D(-z)$$

Now, ϕ_0, the ground state of the oscillator satisfies

$$\langle \phi_0, (\hat{Q} - q\mathbb{1})^2 \phi_0\rangle \langle \phi_0, (\hat{P} - p\mathbb{1})^2 \phi_0\rangle = \frac{1}{4} \quad (p = q = 0)$$

but $\langle z|(\hat{Q} - q\mathbb{1})^2|z\rangle = \langle \phi_0, \left(D(-z)(\hat{Q} - q\mathbb{1})D^{-1}(-z)\right)\left(D(-z)(\hat{Q} - q\mathbb{1})D^{-1}(-z)\right)\phi_0\rangle$

where $D(-z)(\hat{Q} - q\mathbb{1})D^{-1}(-z) = D(-z)\hat{Q}D^{-1}(-z) - q\mathbb{1} = (\hat{Q} + q\mathbb{1}) - q\mathbb{1} = \hat{Q}$

and thus $\langle z|(\hat{Q} - q\mathbb{1})^2|z\rangle = \langle \phi_0, \hat{Q}^2\phi_0\rangle$ and similarly

$\langle z|(\hat{P} - q\mathbb{1})|z\rangle = \langle \phi_0, \hat{P}^2\phi_0\rangle$ and therefore:

$$\langle z|(\hat{Q} - q\mathbb{1})^2|z\rangle \langle z|(\hat{P} - p\mathbb{1})^2|z\rangle = \langle \phi_0, \hat{Q}^2\phi_0\rangle\langle \phi_0, \hat{P}^2\phi_0\rangle = \frac{1}{4}.$$

NOTE: $D(z_1)D(z_2) = e^{\frac{1}{2}(z_1^{*} z_2 - z_1 z_2^{*})} D(z_1 + z_2)$ — purely imaginary! $\Rightarrow$ a phase

— this also follows from: $e^{A} B e^{-A} = B + [A,B] + \dots$

10.1 c) By exercise ~~4+5.~~ 6: $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$

provided $[A,[A,B]] = [B,[A,B]] = 0$.

– since $D(z) = e^{z a^\dagger - \bar{z} a}$ and $[z a^\dagger, -\bar{z} a] = -|z|^2 [a^\dagger, a] = -|z|^2$

we get

$$D(z) = e^{-\frac{|z|^2}{2}} e^{z a^\dagger} e^{-\bar{z} a}$$

– applying this in $|z\rangle = D(z)|\phi_0\rangle$ gives:

$$|z\rangle = D(z)|\phi_0\rangle = e^{-\frac{|z|^2}{2}} e^{z a^\dagger} \Big\{ \mathbb{1} + \sum_{j\geq 1} \frac{\bar{z}^j}{j!} a^j \Big\} |\phi_0\rangle$$

$$= e^{-\frac{|z|^2}{2}} e^{z a^\dagger} |\phi_0\rangle \qquad (a|\phi_0\rangle = 0)$$

$$= e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \frac{z^m}{m!} (a^\dagger)^m |\phi_0\rangle \quad , \quad a^\dagger |\phi_\ell\rangle = \sqrt{\ell+1}\, |\phi_{\ell+1}\rangle$$

$$= e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} |\phi_m\rangle$$

d) $$\langle z_1 | z_2 \rangle = e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)} \sum_m \sum_n \underbrace{\langle \phi_m | \phi_n \rangle}_{=\delta_{mn}} \frac{\bar{z}_1^m z_2^n}{\sqrt{m!\, n!}}$$

$$= \exp\Big[-\frac{1}{2}(|z_1|^2 + |z_2|^2) + \bar{z}_1 z_2 \Big] \qquad (3)$$

To calculate the integral we apply also the representation of c):

$$\int_{\mathbb{C}} |z\rangle\langle z| d^2z = \int_{\mathbb{C}} e^{-|z|^2} \sum_{m,n} |\phi_m\rangle\langle\phi_n| \frac{z^m \bar{z}^n}{\sqrt{m!\, n!}} d^2 z \quad , \quad d^2 z = dx\, dy = r\, dr\, d\theta$$

$$= \int_0^\infty \int_0^{2\pi} e^{-r^2} \sum_{m,n} |\phi_m\rangle\langle\phi_n| \frac{r^{m+n}}{\sqrt{m!\, n!}} e^{i(m-n)\theta}\, r\, d\theta\, dr$$

– but $\int_0^{2\pi} e^{i(m-n)\theta} d\theta = 2\pi \cdot \delta_{mn}$ so we're left:

$$= 2\pi \sum_{m=0}^{\infty} \Big\{ \int_0^\infty \frac{r^{2m+1}}{m!} e^{-r^2} dr \Big\} |\phi_m\rangle\langle\phi_m| \qquad (4)$$

– by setting $s := r^2$ the integral becomes

$$\frac{1}{2} \int_0^\infty \frac{s^m}{m!} e^{-s} ds = \frac{1}{2m!} \int_0^\infty s^m e^{-s} ds = \frac{1}{2m!} m! = \frac{1}{2}$$

$\Rightarrow$ substitute into (3): the result follows.

Finally, $\mathrm{Tr}(A) \neq \int \langle z|A|z\rangle d^2 z$

However, let $A \in B(\mathcal{H}_s)$

$\Rightarrow \exists B, C$ self adjoint: $A = B + iC$. Let $B = \sum_i b_i |b_i\rangle\langle b_i|$, $C = \sum_j c_j |c_j\rangle\langle c_j|$

then $$\int_{\mathbb{C}} \langle z|A|z\rangle d^2 z = \sum_i b_i \int \langle z|b_i\rangle\langle b_i|z\rangle dz + i \sum_j c_j \int \langle z|c_j\rangle\langle c_j|z\rangle dz$$

$$= \sum_i b_i \int \langle b_i|z\rangle\langle z|b_i\rangle dz + \ldots = \pi \cdot \mathrm{Tr}(A).$$

Proofs for exercise 10.2. Before the actual proof let us state the following:

$$\text{If } A^\dagger = A \text{ and } \mathrm{Tr}\left\{A\mathrm{e}^{\lambda_1 a^\dagger}\mathrm{e}^{-\lambda_2^* a}\right\} \;=\; 0 \text{ for all } \lambda_1,\lambda_2 \in \mathbb{C}\,,\ \text{then } A = 0\,. \tag{0.1}$$

Let us assume that this is proven. Let us denote

$$\mathcal{L}\rho \;:=\; -\mathrm{i}\big[\omega a^\dagger a,\rho\big] \;+\; \frac{\gamma}{2}\big(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\big)\,.$$

Then we know from previous exercises that

$$\mathrm{Tr}\left\{(\mathrm{e}^{t\mathcal{L}}\rho)\mathrm{e}^{\lambda_1 a^\dagger}\mathrm{e}^{-\lambda_2^* a}\right\} \;=\; \mathrm{Tr}\left\{\rho\mathrm{e}^{w(t)\lambda_1 a^\dagger}\mathrm{e}^{-w^*(t)\lambda_2^* a}\right\} \quad \text{with} \quad w(t) \;=\; \mathrm{e}^{-(\frac{\gamma}{2}\,-\,\mathrm{i}\omega)t}\,, \tag{0.2}$$

i.e., all time dependence can be absorbed into the single scalar function $w : \mathbb{R}_+ \to \mathbb{C}$. To make use of this we note that:

$$\begin{aligned}\mathrm{Tr}\left\{|z_1\rangle\langle z_2|\mathrm{e}^{\lambda_1 a^\dagger}\mathrm{e}^{-\lambda_2^* a}\right\} \;&=\; \sum_{j=1}^{\infty}\langle s_j|z_1\rangle\langle z_2|\mathrm{e}^{\lambda_1 a^\dagger}\mathrm{e}^{-\lambda_2^* a}|s_j\rangle \\ &=\; \langle \mathrm{e}^{\lambda_1^* a} z_2|\mathrm{e}^{-\lambda_2^* a} z_1\rangle \;=\; (\mathrm{e}^{\lambda_1^* z_2})^*\mathrm{e}^{-\lambda_2^* z_1}\langle z_2|z_1\rangle \\ &=\; \langle z_2|z_1\rangle\cdot \mathrm{e}^{\lambda_1 z_2^*\,-\,\lambda_2^* z_1}\,.\end{aligned}$$

Besides proving the **part (9.2b)** of the exercise, this allows us to exchange multipliers $\alpha_j \in \mathbb{C}$ between $|z_j\rangle$ and λ_j as follows:

$$\begin{aligned}\mathrm{Tr}\left\{|z_1\rangle\langle z_2|\mathrm{e}^{(\alpha_1\lambda_1)a^\dagger}\mathrm{e}^{-(\alpha_2\lambda_2)^* a}\right\} \;&=\; \langle z_2|z_1\rangle\cdot \mathrm{e}^{\lambda_1(\alpha_1^* z_2)^*\,-\,\lambda_2^*(\alpha_2^* z_1)} \\ &=\; \frac{\langle z_2|z_1\rangle}{\langle \alpha_2^* z_2|\alpha_1^* z_1\rangle}\cdot \mathrm{Tr}\left\{|\alpha_2^* z_1\rangle\langle\alpha_1^* z_2|\mathrm{e}^{\lambda_1 a^\dagger}\mathrm{e}^{-\lambda_2^* a}\right\}.\end{aligned} \tag{0.3}$$

Now, by expressing

$$\rho(t) \;:=\; \mathrm{e}^{t\mathcal{L}}\rho(0) \;=\; \int_{\mathbb{C}^2} p(t;z_1,z_2)|z_1\rangle\langle z_2|\mathrm{d}z_1\mathrm{d}z_2\,,$$

where the representation $p(t;z_1,z_2)$ is non-unique, and then applying (0.2) and (0.3) yields

$$\begin{aligned}\mathrm{Tr}\left\{\rho(t)\mathrm{e}^{\lambda_1 a^\dagger}\mathrm{e}^{-\lambda_2^* a}\right\} \;&=\; \mathrm{Tr}\left\{\rho(0)\mathrm{e}^{w(t)\lambda_1 a^\dagger}\mathrm{e}^{-w^*(t)\lambda_2^* a}\right\} \\ &=\; \int_{\mathbb{C}^2} p(0;z_1,z_2)\cdot\mathrm{Tr}\left\{|z_1\rangle\langle z_2|\mathrm{e}^{(w(t)\lambda_1)a^\dagger}\mathrm{e}^{-(w(t)\lambda_2)^* a}\right\}\mathrm{d}z_1\mathrm{d}z_2 \\ &=\; \int_{\mathbb{C}^2}\frac{\langle z_2|z_1\rangle}{\langle w(t)^* z_2|w^*(t)z_1\rangle}p(0;z_1,z_2)\cdot\mathrm{Tr}\left\{|w^*(t)z_1\rangle\langle w^*(t)z_2|\mathrm{e}^{\lambda_1 a^\dagger}\mathrm{e}^{-\lambda_2^* a}\right\}\mathrm{d}z_1\mathrm{d}z_2\,.\end{aligned} \tag{0.4}$$

If we define

$$\phi(t;z) := w^*(t)z \qquad \text{and} \qquad \eta(t;z_1,z_2) \;:=\; \frac{\langle z_2|z_1\rangle}{\langle\phi(t;z_2)|\phi(t;z_1)\rangle}\,, \tag{0.5}$$

and move everything back inside the traces in (0.4) then (0.1) yields

$$\rho(t) \;=\; \int_{\mathbb{C}^2}\eta(t;z_1,z_2)\rho(0;z_1,z_2)\cdot|\phi(t;z_1)\rangle\langle\phi(t;z_2)|\mathrm{d}z_1\mathrm{d}z_2 \tag{0.6}$$

Especially, when $\rho(0) = |z\rangle\langle z|$ then $\rho(0;z_1,z_2) = \delta(z_1 - z)\delta(z - z_2)$ and $\eta(t;z,z) = 1/1 = 1$ and thus we get $\mathrm{e}^{t\mathcal{L}}|z\rangle\langle z| = |\phi(t;z)\rangle\langle\phi(t;z)|$ which proves **part (a)** of the exercise.

Finally, by using $\Re(z_1^* z_2) = \frac{1}{2}(z_1^* z_2 + z_1 z_2^*)$ to calculate the absolute value of $\langle z_1|z_2\rangle = \mathrm{e}^{-\frac{1}{2}(|z_1|^2+|z_2|^2)+z_1^* z_2}$ gives

$$|\langle z_1|z_2\rangle| = \mathrm{e}^{-\frac{1}{2}(|z_1|^2+|z_2|^2-z_1^* z_2+z_1 z_2^*)} = \mathrm{e}^{-\frac{1}{2}|z_1-z_2|^2}.$$

Using this for (0.5) proves the **part (c)** of the exercise:

$$\begin{aligned} |\eta(t;z_1,z_2)| &= \mathrm{e}^{-\frac{1}{2}(1-|w(t)|^2)|z_1-z_2|^2} = \mathrm{e}^{-\frac{1}{2}(1-\mathrm{e}^{-\gamma t})|z_1-z_2|^2} \\ &= \mathrm{e}^{-\frac{\gamma}{2}|z_1-z_2|^2 t+\mathcal{O}(\gamma t)^2}. \end{aligned}$$

This shows that the density operator $\mathrm{e}^{t\mathcal{L}}|\psi\rangle\langle\psi|$, with $|\psi\rangle = c_1|z_1\rangle + c_2|z_2\rangle$, becomes diagonal in the approximate position-momentum eigenspace [1] with a speed that is proportional to the separation of the initial states z_1, z_2 in the classical phase space. Thus information leaks to the environment with a speed that is proportional to the classical separation of the superposed states. For macroscopic distances this speed is huge compared to the damping speed since $q = (2\hbar/\omega)^{1/2}\Re z$ and $p = (2\hbar\omega)^{1/2}\Im z$ so that $|z_1 - z_2|^2 \sim \hbar^{-1}|q_1 - q_2|^2 + \hbar^{-1}|q_1 - q_2|^2$.

Finally, to prove (0.1) we note that by taking derivatives of the left hand side w.r.t. λ_1, λ_2 at $\lambda_1 = \lambda_2 = 0$ one obtains expectations of the form $\mathrm{Tr}\,(A(a^\dagger)^n a^m)$, $n, m \in \mathbb{N}_0$. Now, because any polynomial of Q, or P, can be expressed as a linear combinations of this kind of terms the condition on (0.1) implies $\mathrm{Tr}\,(Ap(Q)) = 0$ for polynomials p. The state of a simple mechanic harmonic oscillator in $\mathbb{R}$ which has no internal degreed of freedom can be represented as $L^2(\mathbb{R},\mu)$ function where $\mu(\mathrm{d}x) = \phi_0(x)\mathrm{d}x$. Since polynomials are dense in this space we see that (0.1) holds. $\square$

[1] To understand this recall the result of Exercise 10.1b. Especially, note that in usual physical units the right hand side becomes $\hbar^2/4$ where $\hbar$ is the Planck constant.