Quantum Probability: Exercise set 9 (C323, Friday 16.4.2010, 16-18) - corrected

Let $H = \int \omega(p) a^\dagger(p) a(p) \mathrm{d}p$ be the Fock space Hamiltonian corresponding the single particle Hamiltonian h on $\mathcal{H}$. Denote the canonical and grand canonical ensembles by

$$\rho_\beta := \frac{\mathrm{e}^{-\beta H}}{\mathrm{Tr}\, \mathrm{e}^{-\beta H}} \qquad \text{and} \qquad \rho_{\beta,\mu} := \frac{\mathrm{e}^{-\beta(H-\mu N)}}{\mathrm{Tr}\, \mathrm{e}^{-\beta(H-\mu N)}},$$

respectively. Here $N := \int a^\dagger(p) a(p) \mathrm{d}p$ is the total number operator and the constant $\mu > 0$ is the chemical potential. The ground state of the field is denoted by Ω.

Denote by $\mathcal{P}_{2m}$ and $\mathcal{S}_n$ the set of all pairings of $\{1, 2, \ldots, 2m\}$ and the set of permutations of $\{1, 2, \ldots, n\}$, respectively. Let $\Phi(f) := a^\dagger(f) + a(f) = \int f(p) a^\dagger(p) + \bar{f}(p) a(p) \mathrm{d}p$ where $f(p) = \langle \phi(p), f \rangle$ and $\phi(p)$ is the eigenfunction of the mode p. As usual, denote operators in the interaction picture by adding a subscript I, e.g., $a_I(p) \equiv a_I(p;t) = \mathrm{e}^{\mathrm{i}tH} a(p) \mathrm{e}^{-\mathrm{i}tH}$, etc.

Hint: In general the proofs for fermions are follow same lines as the corresponding proofs for bosons in the lecture notes.

1. Some details:

   (a) Find the explicit expression for $\sigma(A) \in \{+1, -1\}$ on page 123 of the lecture notes?

   (b) Prove that the number of pairings of $\{1, \ldots, 2m\}$, the number $|\mathcal{P}_{2m}|$, equals $\frac{(2m)!}{2^m m!}$

   (c) For both bosons and fermion show that $\mathrm{e}^{\alpha H} a^\dagger(f) = a^\dagger(\mathrm{e}^{\alpha h} f) \mathrm{e}^{\alpha H}$ and $a(g) \mathrm{e}^{\alpha H} = \mathrm{e}^{\alpha H} a(\mathrm{e}^{\bar{\alpha} h} g)$ for any $\alpha \in \mathbb{C}$, $f, g \in \mathcal{H}$.

2. Zero-temperature case for bosons and fermions:

   (a) Prove by induction on $m \in \mathbb{N}$ that for bosons:

   $$\langle \Omega, \Phi(f_1)\Phi(f_2)\ldots\Phi(f_{2m})\Omega \rangle = \sum_{P \in \mathcal{P}_{2m}} \prod_{(i,j) \in P} \langle f_i, f_j \rangle$$

   (b) Find similar expression for Fermions.

   (c) Prove that one can write

   $$\langle \Omega, \Phi_I(f_1; t_1) \cdots \Phi_I(f_{2m}; t_{2m})\Omega \rangle = \sum_{P \in \mathcal{P}_{2m}} \prod_{(i,j) \in P} h(t_i - t_j; f_i, f_j),$$

   for fermions. What are the functions h?

3. Finite temperature case for fermions:

   (a) Show that the pair correlation function is

   $$\mathrm{Tr}\left\{a^\dagger(f) a(g) \rho_{\beta,\mu}\right\} = \left\langle g, \tfrac{1}{1+\mathrm{e}^{\beta(h-\mu)}} f \right\rangle$$

   (b) Show that

   $$\mathrm{Tr}\left\{a^\dagger(f_1) \cdots a^\dagger(f_n) a(g_n) \cdots a(g_1) \rho_\beta\right\} = \sum_{\pi \in S_n} (-1)^\pi \prod_{i=1}^{n} \left\langle g_i, \tfrac{1}{1+\mathrm{e}^{\beta(h-\mu)}} f_{\pi(i)} \right\rangle$$

4. The generating function: Prove that for bosons one gets:

   $$\mathrm{Tr}\left\{\mathrm{e}^{\mathrm{i}\Phi(f)} \rho_\beta\right\} = \exp\left[-\tfrac{1}{2}\left\langle f, \tfrac{1+\mathrm{e}^{-\beta h}}{1-\mathrm{e}^{-\beta h}} f \right\rangle\right].$$

(9.1)

(a) $\sigma(A)$ is just the coefficient in a general expression:

$$[X_1,[X_2,\dots,[X_m,Y]\dots]] = \sum_{A\subset\{1,2,\dots,m\}} \sigma(A) \underbrace{X_{i_1}X_{i_2}\cdots X_{i_m}}_{=:X_A} Y \underbrace{X_{j_1}X_{j_2}\cdots X_{j_{M-m}}}_{=:Y_{A^c}}$$

where $A^c := \{1,2,\dots,M\}\setminus A$
and $A=\{i_1,\dots,i_m\}$ and $A^c=\{j_1,j_2,\dots,j_{M-m}\}$
and $i_l < i_{l+1}$ and $j_l > j_{l+1}$.

- Since $[X,Y] = (+X)Y + Y(-X)$ one sees that

$$\sigma(A) = (-1)^{|A^c|} = (-1)^{M-|A|}$$

- This can be proven formally by induction on $M\in\mathbb{N}$.

(b) - Let's recall: suppose there are m objects and we ask how many ways one can put m_1 of these into 1:st "box" and m_2 into 2-nd "box", and so on, such that $m_1+m_2+\dots+m_l = m$, i.e., there are l boxes in total.

- The answer is

$$\frac{m\cdot(m-1)\cdots(m+1-m_1)}{m_1!} \times \frac{(m-m_1)\cdots(m+1-m_1-m_2)}{m_2!} \times\dots\times \frac{m_l\cdot(m_l-1)\cdots 2\cdot 1}{m_l!}$$

$$= \frac{m!}{\prod_{i=1}^{l} m_i!} \equiv \binom{m}{m_1\, m_2 \cdots m_l}$$

- This implies that there are

$$\binom{2m}{\underset{(1)}{2}\,\underset{(2)}{2}\cdots \underset{(m)}{2}} = \frac{(2m)!}{2^m}$$

ways to choose m <u>labeled</u> pairs.

- Since for us the pairs are not labeled, e.g., $(1,3),(2,4)$ is considered same as $(2,4),(1,3)$ we must further divide by $m!$

$\Rightarrow$ we get $|\mathcal{P}_{2m}| = \frac{(2m)!}{2^m m!}$ ⊠

(c) Let $S f_1\otimes\dots\otimes f_m = \frac{1}{m!}\sum_{\pi\in S_m} f_{\pi(1)}\otimes\dots\otimes f_{\pi(m)}$

$\mathcal{A} f_1\otimes\dots\otimes f_m = \frac{1}{m!}\sum_{\pi\in S_m} (-1)^\pi f_{\pi(1)}\otimes\dots\otimes f_{\pi(m)}$

By exercise (6.1) we have

$$e^{\alpha H} f_1\otimes\dots\otimes f_m = e^{\alpha h} f_1 \otimes\dots\otimes e^{\alpha h} f_m$$

Thus

$$a(g)\, e^{\alpha H} \mathcal{A} f_1\otimes\dots\otimes f_m = \frac{1}{m!}\sum_{\pi\in S_m} (-1)^\pi a(g)\, e^{\alpha h} f_{\pi(1)}\otimes\dots\otimes e^{\alpha h} f_{\pi(m)}$$

$$= \frac{\sqrt{m}}{m!}\sum_{\pi\in S_m} (-1)^\pi \langle g, e^{\alpha h} f_{\pi(1)}\rangle\, e^{\alpha h} f_{\pi(2)}\otimes\dots\otimes e^{\alpha h} f_{\pi(m)}$$

$= (1)$

9.1c continues...

- Similarly:

$$e^{\alpha H} a(\tilde{y}) \mathcal{A} f_1 \otimes \cdots \otimes f_m = \frac{\sqrt{m}}{m!} \sum_{\pi \in S_m} (-1)^\pi e^{\alpha H} \langle \tilde{y}, f_{\pi(1)} \rangle f_{\pi(2)} \otimes \cdots \otimes f_{\pi(m)}$$

$$= \frac{\sqrt{m}}{m!} \sum_{\pi \in S_m} (-1)^\pi \langle \tilde{y}, f_{\pi(1)} \rangle e^{\alpha h} f_{\pi(2)} \otimes \cdots \otimes e^{\alpha h} f_{\pi(m)}$$

- So, if we choose $= (2)$

$$\tilde{y} = e^{\bar{\alpha} h} y \quad \text{then} \quad \langle \tilde{y}, f_{\pi(1)} \rangle = \langle e^{\bar{\alpha} h} y, f_{\pi(1)} \rangle \overset{(e^{\bar{\alpha} h})^\dagger = e^{\alpha h}}{=} \langle y, e^{\alpha h} f_{\pi(1)} \rangle$$

and thus (1) = (2). Moreover, vectors of the form

$$\mathcal{A} f_1 \otimes \cdots \otimes f_m, \quad m \in \mathbb{N}$$

are dense in the Fock-space $\Rightarrow$ (1) = (2) implies the claim.

- Now, bosons are treated similarly: replace $\mathcal{A}$ by $\mathcal{S}$ and $(-1)^\pi$ by 1.

- Moreover,

$$e^{\alpha H} a^\dagger(y) = \left(a(y) e^{\bar{\alpha} H} \right)^\dagger$$

$$= \left(e^{\bar{\alpha} H} a(e^{\alpha h} y) \right)^\dagger$$

$$= a^\dagger(e^{\alpha h} y) e^{\alpha H}$$

since $(a(f))^\dagger = a^\dagger(f)$.

(9.2) Let's prove (a) and (b) at the same time:

- Denote: $\Phi_i := \Phi(f_i)$ and $a_i := a(f_i)$ and

$$\Phi_{i:j} := \Phi_i \Phi_{i+1} \cdots \Phi_{j-1}\Phi_j = \Phi(f_i)\cdots\Phi(f_j) \quad \text{for } i \le j.$$

- For bosons (+) and fermions (−) we have:

$$a_i a_j^\dagger = \pm a_j^\dagger a_i + \langle f_i, f_j\rangle$$

So: (Let's calculate with fermions - the case of bosons follows by replacing all minus signs with "+")

$$\langle \Omega, \Phi_{1:m}\Omega\rangle = \langle \Omega, (a_1\Phi_2)\Phi_{3:m}\Omega\rangle$$
$$= \langle \Omega, (-\Phi_2 a_1 + \langle f_1, f_2\rangle)\Phi_{3:m}\Omega\rangle$$
$$= \langle f_1, f_2\rangle\langle \Omega, \Phi_{3:m}\Omega\rangle - \langle \Omega, \Phi_2(a_1\Phi_3)\Phi_{4:m}\Omega\rangle$$

- similarly we get:

$$\langle \Omega, \Phi_{2:i-1}(a_1\Phi_i)\Phi_{i+1:m}\Omega\rangle = \langle f_1, f_i\rangle\langle \Omega, \Phi_{2:i-1}\Phi_{i+1:m}\Omega\rangle - \langle \Omega, \Phi_{2:i}(a_1\Phi_{i+1})\Phi_{i+2:m}\Omega\rangle$$

so that by iterating:

$$\langle \Omega, \Phi_{1:m}\Omega\rangle = \sum_{j=2}^{m} (-1)^j \langle f_1, f_j\rangle \langle \Omega, \Phi_{2:j-1}\Phi_{j+1:m}\Omega\rangle \qquad (1)$$

since $a_1\Omega = 0$. Now, if we write $(-1)^j = (-1)^{j-1-1}$ then we can identify the exponent $(j-1-1)$ as the number of basic swaps of type $(i_1 i_2 \cdots i_{j-1} i_j i_{j+1} i_{j+2} \cdots i_m) \mapsto (i_1 i_2 \cdots i_{j-2} i_{j+1} i_j i_{j+2} \cdots i_m)$ to bring 1:st and j:th element next to each others.

- Now, let's show that (1) yields an expansion:

$$\langle \Omega, \Phi_{1:m}\Omega\rangle = \sum_{P\in\mathcal{P}_m} (-1)^P \prod_{(i,j)\in P} \langle f_i, f_j\rangle \qquad (2)$$

where $(-1)^P := (-1)^{\#\{\text{swaps needed to bring } 12\cdots m \text{ to } i_1 j_1 i_2 j_2 \cdots i_{m/2} j_{m/2}\}}$

$$\equiv (-1)^{\pi(P)}$$

where $\pi(P) \in S_m$ is a (non-unique) permutation that satisfies: $(12\cdots m) \overset{\pi}{\longmapsto} (i_1 j_1 i_2 j_2 \cdots i_{m/2} j_{m/2})$ (3)

with $i_\ell < j_\ell$ and $(i_\ell, j_\ell)\in P$, $\forall \ell = 1 \ldots m/2$.

- <u>Note:</u> It's easy to see that the order of the pairs does not matter, i.e., if $\pi, \tilde{\pi} \in S_m$ both satisfy (3) then $(-1)^{\tilde{\pi}} = (-1)^{\pi}$: $(i_1 j_1 i_2 j_2) \sim -(i_1 i_2 j_1 j_2) \sim (i_2 i_1 j_1 j_2) \sim \ldots \sim (i_2 j_2 i_1 j_1)$

- To prove (3) we simply note that (1) is the needed induction step while the claim trivially holds for $m=2$.

(9.2) continues...

c) We have $m = 2M$:

$$\textcircled{A} := \langle \Omega, \Phi_I(f_1; t_1) \cdots \Phi_I(f_m; t_m)\, \Omega \rangle$$

Now, $\Phi_I(f; t) = e^{itH}\, \Phi(f)\, e^{-itH}$

$= e^{itH} \{ a^\dagger(f) + a(f) \} e^{-itH}$

$= a^\dagger(e^{ith} f) + a(e^{ith} f)$

Where the last comes from 9.1 c)

But this implies $\Phi_I(f; t) = \Phi(e^{ith} f)$

And thus (b) implies:

$$\textcircled{A} = \sum_{P \in \mathcal{P}_m} (-1)^P \prod_{(i,j) \in P} \underbrace{\langle e^{it_i h} f_i, e^{it_j h} f_j \rangle}_{\langle f_i, e^{-i(t_i - t_j)h} f_j \rangle \,=:\, h(t_i - t_j; f_i, f_j)}$$

(9.3) a) Since $\rho_{\beta,N} = \dfrac{e^{-\beta(H-\mu N)}}{\mathrm{Tr}\, e^{-\beta(H-\mu N)}}$ by setting $Z = \mathrm{Tr}(e^{-\beta(H-\mu N)})$ we get:

$$(A) := \mathrm{Tr}\left[\rho_{\beta,N}\, a^\dagger(f) a(g)\right] = Z^{-1} \cdot \mathrm{Tr}\left[e^{-2\beta\tilde{H}} a^\dagger(f) a(g)\right]$$

where $\tilde{H} = \frac{1}{2}(H - \mu N)$

$$= \frac{1}{2}\int (\omega(p) - \mu)\, a^\dagger(p) a(p)\, dp$$

$$= \int \underbrace{\left(\frac{\omega(p)-\mu}{2}\right)}_{=:\,\tilde{\omega}(p)} a^\dagger(p) a(p)\, dp$$

Exercise (9.1) then implies:

$$(1) \quad \begin{aligned} e^{-\beta\tilde{H}} a^\dagger(f) &= a^\dagger(e^{-\beta\tilde{h}} f)\, e^{-\beta\tilde{H}} \\ a(g)\, e^{-\beta\tilde{H}} &= e^{-\beta\tilde{H}} a(e^{-\beta\tilde{h}} g) \end{aligned}$$

where $\tilde{h}$ is a single particle hamiltonian:

$$\tilde{h} = \int \tilde{\omega}(p)\, P(dp)$$

with $P(dp)$ formally equal to $|\phi(p)\rangle\langle\phi(p)|\, dp$, $\tilde{h}\,\phi(p) = \tilde{\omega}(p)\,\phi(p)$.

- Now, using (1) to manipulate (A) we get:

$$(A) = Z^{-1} \cdot \mathrm{Tr}\left[(e^{-\beta\tilde{H}} a^\dagger(f))(a(g)\, e^{-\beta\tilde{H}})\right]$$

$$= Z^{-1} \cdot \mathrm{Tr}\left[(a^\dagger(e^{-\beta\tilde{h}} f)\, e^{-\beta\tilde{H}})(e^{-\beta\tilde{H}} a(e^{-\beta\tilde{h}} g))\right]$$

$$= \mathrm{Tr}\left[a(e^{-\beta\tilde{h}} g)\, a^\dagger(e^{-\beta\tilde{h}} f)\, \rho_{\beta,N}\right]$$

$(e^{-\beta\tilde{h}})^\dagger = e^{-\beta\tilde{h}}$

$$= \mathrm{Tr}\left[\left\{-a^\dagger(e^{-\beta\tilde{h}} f)\, a(e^{-\beta\tilde{h}} g) + \langle e^{-\beta\tilde{h}} g, e^{-\beta\tilde{h}} f\rangle\right\} \rho_{\beta,N}\right]$$

$$= \langle g, e^{-2\beta\tilde{h}} f\rangle - \mathrm{Tr}\left[a^\dagger(e^{-\beta\tilde{h}} f)\, a(e^{-\beta\tilde{h}} g)\, \rho_{\beta,N}\right]$$

- The second term is of the same form as (A) except $f \mapsto e^{-\beta\tilde{h}} f$, $g \mapsto e^{-\beta\tilde{h}} g$

$\Rightarrow$ iterate:

$$(A) = \sum_{j=1}^{M} (-1)^{j-1} \langle g, (e^{-2\beta\tilde{h}})^j f\rangle + (-1)^M \mathrm{Tr}\left[a^\dagger(e^{-\beta M\tilde{h}} f)\, a(e^{-\beta M\tilde{h}} f)\, \rho_{\beta,N}\right]$$

- Now, the remainder term $\to 0$ as $M \to \infty$ since $\tilde{h} > 0$ (*)

- Thus we have

$$\mathrm{Tr}\left[a^\dagger(f) a(g)\, \rho_{\beta,N}\right] = \left\langle g, e^{-2\beta\tilde{h}} \sum_{j=0}^{\infty} (-e^{-2\beta\tilde{h}})^j f\right\rangle$$

$$= \left\langle g, \frac{e^{-2\beta\tilde{h}}}{1 + e^{-2\beta\tilde{h}}} f\right\rangle$$

$$= \left\langle g, \frac{1}{1 + e^{\beta(h-\mu)}} f\right\rangle$$

9.3. b) WRITE $a^\dagger(f_{i:j}) := a^\dagger(f_i)\, a^\dagger(f_{i+1}) \cdots a^\dagger(f_j)$ AND SIMILARLY FOR $a(g_{i:j})$ FOR $i \le j$. THEN:

$$(A) := \mathrm{Tr}\left[a^\dagger(f_{1:m})\, a(g_{1:m})\, \rho_{\beta,N} \right] = Z \cdot \mathrm{Tr}\left[a^\dagger(f_{1:m})\, a(g_{1:m-1}) \left(a(g_m)\, e^{-\beta(H-\nu N)} \right) \right]$$

$$= Z \cdot \mathrm{Tr}\left[a^\dagger(f_{1:m})\, a(g_{1:m-1}) \left(e^{-\beta(H-\nu N)}\, a(w\cdot g_m) \right) \right]$$

$$= \mathrm{Tr}\left[\left(a(w\cdot g_m)\, a^\dagger(f_1) \right) a^\dagger(f_{2:m})\, a(g_{1:m-1})\, \rho_{\beta,N} \right] \qquad (*).$$

where $Z := \mathrm{Tr}\, e^{-\beta(H-\nu N)} \in \mathbb{R}_+$, AND $w(p) = e^{-\beta(\omega(p)-\nu)}$.

NOW, WE JUST USE $a(f)\, a^\dagger(g) = -a^\dagger(g)\, a(f) + \langle f, g\rangle$:

$$(A) = \mathrm{Tr}\left[\left\{ -a^\dagger(f_1)\, a(w\cdot g_m) + \langle w\cdot g_m, f_1 \rangle \right\} \cdot a^\dagger(f_{2:m})\, a(g_{1:m-1})\, \rho_{\beta,N} \right]$$

$$= \langle g_m, w\cdot f_1 \rangle\, \mathrm{Tr}\left[a^\dagger(f_{2:m})\, a(g_{1:m-1})\, \rho_{\beta,N} \right]$$

$$- \mathrm{Tr}\left[a^\dagger(f_1)\, a(w g_m)\, a^\dagger(f_{2:m})\, a(g_{1:m-1})\, \rho_{\beta,N} \right]$$

BY MOVING $a(w g_m)$ ACROSS $a^\dagger(f_{2:m})$ IN THE SECOND TERM YIELDS:

$$(A) = \sum_{j=1}^{m} (-1)^{j-1} \langle g_m, w f_j \rangle\, \mathrm{Tr}\left[a^\dagger(f_{1:j-1})\, a^\dagger(f_{j+1:m})\, a(g_{1:m-1})\, \rho_{\beta,N} \right]$$

$$+ (-1)^m\, \mathrm{Tr}\left[a^\dagger(f_{1:m})\, a(w g_m)\, a(g_{1:m-1})\, \rho_{\beta,N} \right]$$

NOW, IF WE MOVE $a(w g_m)$ PAST $a(g_{1:m-1})$ WE GET EXTRA FACTOR $(-1)^{m-1}$. BY MOVING $a(w g_m)$ PAST $\rho_{\beta,N}$ AS WELL GIVES:

$$(A) = \sum_{j=1}^{m} (-1)^{j-1} \langle g_m, w f_j \rangle\, \mathrm{Tr}\left[a^\dagger(f_{1:j-1})\, a^\dagger(f_{j+1:m})\, a(g_{1:m-1})\, \rho_{\beta,N} \right]$$

$$- \mathrm{Tr}\left[a(w^2 g_m)\, a^\dagger(f_{1:m})\, a(g_{1:m-1})\, \rho_{\beta,N} \right]$$

$= (-1)^m (-1)^{m-1}$

BUT NOW, THE LAST TERM EQUALS $(*)$ WITH $w \mapsto w^2$.

ITERATING AND USING THE POSITIVITY OF $H-\nu N$ TO ARGUE $w^m \to 0$ AS $m \to \infty$ WE GET:

$$(A) = \sum_{j=1}^{m} (-1)^{j-1} \sum_{l=1}^{\infty} (-1)^{l-1} \langle g_m, w^l f_j \rangle\, \mathrm{Tr}\left[a^\dagger(f_{1:j-1})\, a^\dagger(f_{j+1:m})\, a(g_{1:m-1})\, \rho_{\beta,N} \right]$$

$$= \sum_{j=1}^{m} (-1)^{j-1} \left\langle g_m, \frac{w}{1+w} f_j \right\rangle \cdot \mathrm{Tr}\left[a^\dagger(f_{1:j-1})\, a^\dagger(f_{j+1:m})\, a(g_{1:m-1})\, \rho_{\beta,N} \right]$$

(9.4) – From notes: $M=2m$:

(0) $$\mathrm{Tr}\{\Phi(f_1)\cdots\Phi(f_n)\rho_\beta\} = \sum_{P\in\mathcal{P}_M} \prod_{(i,j)\in P} G_\beta(f_i,f_j)$$

with $G_\beta(f,g) := \mathrm{Tr}\{\Phi(f)\Phi(g)\rho_\beta\}$. For $M=2m+1$ LHS of (0) yields 0.

– Now,

(1) $$\mathrm{Tr}\{e^{i\Phi(f)}\rho_\beta\} = \sum_{M=0}^{\infty} \frac{i^M}{M!}\mathrm{Tr}\{\Phi(f)^M\rho_\beta\}$$

– by using (0) one get for $M=2\ell+1$ zero and $M=2\ell$:

$$\mathrm{Tr}\{\Phi(f)^{2\ell}\rho_\beta\} = |\mathcal{P}_{2\ell}|\cdot G_\beta(f,f)^\ell = \frac{(2\ell)!}{\ell!\,2^\ell}\left\langle f, \frac{1+e^{-\beta h}}{1-e^{-\beta h}} f\right\rangle^\ell$$

– Thus (1) becomes

$$\mathrm{Tr}\{e^{i\Phi(f)}\rho_\beta\} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell)!}\frac{(2\ell)!}{\ell!}\left(\frac{1}{2}\left\langle f, \frac{1+e^{-\beta h}}{1-e^{-\beta h}} f\right\rangle\right)^\ell = \exp\left[-\frac{1}{2}\left\langle f, \frac{1+e^{-\beta h}}{1-e^{-\beta h}} f\right\rangle\right] \quad \square$$