Quantum Probability: Solution to Exercise 8.1

1. *Let $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$, with $\dim(\mathcal{H}) = n$, be a completely positive superoperator. Show that two Krauss representations*

$$\Phi(\rho) \;=\; \sum_{\alpha=1}^{a} M_\alpha \rho M_\alpha^\dagger \;=\; \sum_{\beta=1}^{b} N_\alpha \rho N_\alpha^\dagger\,,$$

are related by an unitary matrix $U \in \mathbb{C}^{n^2\times n^2}$ so that

$$M_\alpha = \sum_{\beta=1}^{n^2} U_{\alpha\beta} N_\beta\,,$$

where $M_\alpha = N_\beta = 0$ for $\alpha \geq a+1$ and $\beta \geq b+1$. Since the dimension of unitary matrices in $\mathbb{C}^n$ is n^2 this shows that there can not be more than n^4 Krauss representations for Φ.

Let $\{s_j\}$ and $\{e_j\}$ be arbitrary but fixed ON-bases of $\mathcal{H}_S$ and $\mathcal{H}_E$ respectively. Define and unnormalized maximally entangled vector:

$$\Psi \;:=\; \sum_i |s_i, e_i\rangle$$

By using the two Krauss decompositions one gets

$$\rho_{SE} \;:=\; (\Phi\otimes 1_E)|\Psi\rangle\langle\Psi| \;=\; \sum_{\alpha=1}^{a} |\tilde{\Psi}_\alpha\rangle\langle\tilde{\Psi}_\alpha| \;=\; \sum_{\alpha=1}^{b} |\tilde{\Omega}_\beta\rangle\langle\tilde{\Omega}_\beta|\,,$$

where $\tilde{\Psi}_\alpha, \tilde{\Omega}_\beta \in \mathcal{H}_S\otimes\mathcal{H}_E$ are defined by

$$\tilde{\Psi}_\alpha \;:=\; (M_\alpha\otimes 1_E)\Psi \qquad \text{and} \qquad \tilde{\Omega}_\beta \;:=\; (N_\beta\otimes 1_E)\Psi\,.$$

These are not assumed to be normalized nor even linearly independent.

Now, there is the general trick: Any operator A acting on $\mathcal{H}_S$ satisfies

$$A\phi = (1_S\otimes\langle\phi^*|)(A\otimes 1_E)\Psi\,,$$

where $\phi^* := \sum_i \phi_i^* e_i \in \mathcal{H}_E$ for $\phi = \sum_i \phi_i s_i \in \mathcal{H}_S$. We will use the this trick to write

$$M_\alpha\phi \;=\; (1_S\otimes\langle\phi^*|)\tilde{\Psi}_\alpha \qquad \text{and} \qquad N_\beta \;=\; (1_S\otimes\langle\phi^*|)\tilde{\Omega}_\beta\,. \tag{0.1}$$

Let us purify these be in terms of the vectors in $\mathcal{H}_S\otimes\mathcal{H}_E\otimes\mathcal{H}_R$:

$$\mathcal{A} \;:=\; \sum_{\alpha=1}^{a} \tilde{\Psi}_\alpha\otimes r_\alpha \qquad \text{and} \qquad \mathcal{B} \;:=\; \sum_{\beta=1}^{b} \tilde{\Omega}_\beta\otimes r_\beta\,,$$

Here $\{r_\gamma\}$ is a ON-basis for a Hilbert space $\mathcal{H}_R$. Since $a, b \leq n^2$ we may take $\dim(\mathcal{H}_R) = n^2$ and define $|\tilde{\Psi}_\alpha\rangle = |\tilde{\Omega}_\beta\rangle = 0$ for $\alpha \geq a+1$ and $\beta \geq b+1$. Since $\rho_{SE} = \mathrm{Tr}_R(|\mathcal{A}\rangle\langle\mathcal{A}|) = \mathrm{Tr}_R(|\mathcal{B}\rangle\langle\mathcal{B}|)$ there exists an unitary transformation U acting on $\mathcal{H}_R$ such that

$$\mathcal{A} \;=\; (1_{SE}\otimes U)\mathcal{B}\,,$$

where $1_{SE} := 1_S \otimes 1_E$. Now expressing $U = \sum_{\alpha,\beta} U_{\alpha\beta} |r_\alpha\rangle\langle r_\beta|$ one gets

$$\begin{aligned}\sum_\alpha \tilde{\Psi}_\alpha \otimes r_\alpha \;&=\; \mathcal{A} \;=\; (1_{SE} \otimes U)\mathcal{B} \;=\; (1_{SE} \otimes U) \sum_\beta \tilde{\Omega}_\beta \otimes r_\beta \\ &=\; \sum_\alpha \Big\{ \sum_\beta U_{\alpha\beta}\, \tilde{\Omega}_\beta \Big\} \otimes r_\alpha \,.\end{aligned}$$

By the ON property of $\{r_\alpha\}$ each α term above is ortogonal to the others. Thus we get

$$\tilde{\Psi}_\alpha \;=\; \sum_\beta U_{\alpha\beta}\, \tilde{\Omega}_\beta \qquad \text{for each} \qquad \alpha = 1, 2, \ldots, n^2$$

By using the representation trick, i.e., (0.1), this implies

$$M_\alpha \phi \;=\; (1_S \otimes \langle \phi^* |) \tilde{\Psi}_\alpha \;=\; \sum_\beta U_{\alpha\beta}\, (1_S \otimes \langle \phi^* |) \tilde{\Omega}_\beta \;=\; \sum_\beta U_{\alpha\beta}\, N_\beta \phi \,.$$

Since $|\phi\rangle \in \mathcal{H}_S$ was arbitrary this completes the proof.

(8.2)

(a) The basis $\{|l\rangle\}$ spans $\mathcal{H}_S$ $\Rightarrow$ operators fully characterized on how they act on the vectors $|l\rangle$:

$$a_I|l\rangle = e^{i\omega t n} a\, e^{-i\omega t n}|l\rangle \quad \text{| NUMBER OPERATOR } n|l\rangle = l|l\rangle$$
$$= e^{i\cdots} a\, e^{-i\omega t l}|l\rangle$$
$$= e^{-i\omega t l} e^{i\omega t n} a|l\rangle$$
$$= e^{-i\omega t l} e^{i\omega t n}\sqrt{l}\,|l-1\rangle = \sqrt{l}\, e^{-i\omega t l} e^{i\omega t(l-1)}|l-1\rangle$$
$$= e^{-i\omega t} a|l\rangle \quad , \forall l \in \mathbb{N}_0$$

$$\Rightarrow \quad a_I(t) = e^{-i\omega t} a$$

(b) - Let's denote $U_0 \equiv U_0(t) := e^{-i\omega t H_0}$

$\Rightarrow \quad i\partial_t U_0 = H_0 U_0$ (fundamental solution to Schrödinger's Eq.)

- Now, $\rho_I \equiv \rho_I(t) = U_0^{-1}(t)\,\rho(t)\,U_0(t)$

or $\rho(t) = U_0(t)\,\rho_I(t)\,U_0^{-1}(t)$

- Let's substitute this into (0.1): LHS gives:

$$\dot{\rho} \equiv \partial_t \rho = \partial_t(U_0 \rho_I U_0^{-1}) = \underbrace{\dot{U}_0 \rho_I U_0^{-1}}_{= -iH_0U_0\rho_I U_0^{-1} = -iH_0\rho} + \underbrace{U_0\rho_I(\partial_t U_0^{-1})}_{\partial_t U_0^{-1} = \partial_t U_0^{\dagger} = (\partial_t U_0)^{\dagger} = +iH_0U^{\dagger}\;\Rightarrow\; iU_0\rho_I U_0^{-1}H_0 = i\rho H_0} + \underbrace{U_0\dot{\rho}_I U_0^{-1}}_{\text{The rest.}}$$

$$= -i[H_0,\rho] + U_0\dot{\rho}_I U_0^{\dagger}$$

- Now, RHS of 0.1 gives:

$$\dot{\rho} = -i[H_0,\rho]$$
$$+ \gamma a\rho a^{\dagger} - \frac{\gamma}{2}a^{\dagger}a\rho - \frac{\gamma}{2}\rho a^{\dagger}a$$

- Comparing LHS and RHS we get:

$$\dot{\rho}_I = U_0^{-1}\left\{a\rho a^{\dagger} - \frac{\gamma}{2}a^{\dagger}a\rho - \frac{\gamma}{2}\rho a^{\dagger}a\right\}U_0$$

- The first term goes:

$$U_0^{-1}a\rho a^{\dagger}U_0 = (U_0^{-1}aU_0)(U_0^{-1}\rho U_0)(U_0^{-1}aU_0)^{\dagger}$$
$$= a_I\,\rho_I\,a_I^{\dagger}$$
$$\overset{(a)}{=} a\,\rho_I\,a^{\dagger}$$

- Other terms go similarly. $\Rightarrow$ $\dot{\rho}_I = \gamma\left\{a\rho_I a^{\dagger} - \frac{1}{2}(a^{\dagger}a\rho_I + \rho_I a^{\dagger}a)\right\}$

8.2 CONTINUES...

(c) Let's define $\tilde{a}(t) := U_0(t)\, a\, U_0^{-1}(t)$ USE: $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$

THEN $f(t) := \mathrm{Tr}\left[\tilde a(t)\,\rho(t)\right] = \mathrm{Tr}\left[\tilde a\, U_0\, \rho_I\, U_0^{-1}\right]$

$= \mathrm{Tr}\left[U_0^{-1}\tilde a\, U_0\, \rho_I\right] = \mathrm{Tr}\left[a\, \rho_I(t)\right]$

- Now, using $[a,a^\dagger]=\mathbb{1}$ and cyclic identity

$$\frac{d}{dt} f(t) = \mathrm{Tr}\{a\,\dot\rho_I\}$$

$$= \frac{\gamma}{2}\mathrm{Tr}\{2a^2\rho a^\dagger - a a^\dagger a\rho - a\rho a^\dagger a\}$$

$$= \frac{\gamma}{2}\left\{2\mathrm{Tr}[a^2\rho a^\dagger] - \mathrm{Tr}[a a^\dagger a\rho] - \mathrm{Tr}[a^2\rho a^\dagger]\right\}$$

$$= \frac{\gamma}{2}\left\{\cancel{2}\,\mathrm{Tr}[a^\dagger a\, a\rho] - \mathrm{Tr}[a a^\dagger a\rho] - \cancel{\mathrm{Tr}[a^\dagger a\, a\rho]}\right\}$$

$$= \frac{\gamma}{2}\mathrm{Tr}\Big(\underbrace{[a^\dagger,a]}_{=-\mathbb{1}}\, a\rho\Big) = -\frac{\gamma}{2}\mathrm{Tr}(a\rho_I)$$

$$= -\frac{\gamma}{2} f(t)$$

- Clearly, this is solved by $f(t) = e^{-\frac{\gamma}{2}t} f(0)$ ⊠

(d) define $g(t) := \mathrm{Tr}(\tilde a^\dagger \tilde a\, \rho(t))$

$$= \mathrm{Tr}(\tilde a^\dagger \tilde a\, U_0\rho_I U_0^{-1}) = \mathrm{Tr}\Big(\big(\underbrace{U_0^{-1}\tilde a^\dagger U_0}_{a^\dagger}\big)\big(\underbrace{U_0^{-1}\tilde a\, U_0}_{a}\big)\,\rho_I\Big)$$

$$= \mathrm{Tr}(a^\dagger a\,\rho_I)$$

THEN $g'(t) = \frac{\gamma}{2}\mathrm{Tr}\left(a^\dagger a\left\{2a\rho_I a^\dagger - a^\dagger a\rho_I - \rho_I a^\dagger a\right\}\right)$

$$= \frac{\gamma}{2}\cdot\left\{2\,\mathrm{Tr}(a^\dagger a^\dagger a a\,\rho_I) - 2\,\mathrm{Tr}(a^\dagger a a^\dagger a\,\rho_I)\right\}$$

$$= \gamma\,\mathrm{Tr}\Big(a^\dagger \underbrace{[a^\dagger,a]}_{=-\mathbb{1}}\, a\,\rho_I\Big) = -\gamma\,\mathrm{Tr}(a^\dagger a\,\rho_I)$$

$$= -\gamma\, g(t)$$

- This is solved by $g(t) = e^{-\gamma t} g(0)$ ⊠

(8.3)

a) Let's start by opening definitions:

(0) $\eta_I(a,\bar{a},t) := \mathrm{Tr}\left(\rho_I(t)\, e^{a a^\dagger} e^{-\bar{a} a}\right)$, $a,\bar{a}\in\mathbb{C}$ (we take $\bar{a}=a^*$ later)

(1) $\partial_t \rho_I = \frac{\gamma}{2}\left(2 a \rho_I a^\dagger - a^\dagger a \rho_I - \rho_I a^\dagger a\right)$

- Let's express $\partial_t \eta_I$ with these by using $\mathrm{Tr}(ABC)=\mathrm{Tr}(CAB)=\mathrm{Tr}(BCA)$:

$$(2)\quad \frac{2}{\gamma}\cdot\partial_t\eta_I = \underbrace{2\mathrm{Tr}\left(\rho_I a^\dagger e^{a a^\dagger} a e^{-\bar{a}a}\right)}_{=(2a)} \underbrace{-\mathrm{Tr}\left(\rho_I e^{a a^\dagger} e^{-\bar{a}a} a^\dagger a\right)}_{=:(2b)} \underbrace{-\mathrm{Tr}\left(\rho_I a^\dagger a e^{a a^\dagger} e^{-\bar{a}a}\right)}_{=:(2c)}$$

- On the other hand:

(3) $\frac{\partial}{\partial a}\eta_I = \mathrm{Tr}\left(\rho_I a^\dagger e^{a a^\dagger} e^{-\bar{a}a}\right)$

(4) $\frac{\partial}{\partial \bar{a}}\eta_I = \mathrm{Tr}\left(\rho_I e^{a a^\dagger}(-a)e^{-\bar{a}a}\right)$

- Our strategy to derive the PDE for η_I is now to bring all terms (2a,b,c), (3), (4) into the standard form:

(5) $\mathrm{Tr}\left(\rho_I (a^\dagger)^n e^{a a^\dagger}(-a)^m e^{-\bar{a}a}\right)$, $n,m\in\mathbb{N}_0\equiv\{0,1,2,3,\ldots\}$

by using the formula:

(6) $e^{tA}Be^{-tA} = B + [A,B]t + f([A,[A,B]],t)$

where $f(0,t)=0$.

- Terms (2a), (3) and (4) are already in the std. form.
- Let's put (2b) into form (5):

$$e^{-\bar{a}a}a^\dagger = \left(e^{-\bar{a}a}a^\dagger e^{\bar{a}a}\right)e^{-\bar{a}a} \qquad \left(\underline{\text{Note:}}\ e^{tA}e^{-tA}=\mathbb{1}\ \text{by power series expansion}\right)$$

$$= \left\{a^\dagger + [a,a^\dagger](-\bar{a}) + 0\right\} e^{-\bar{a}a} \qquad [x,\underbrace{[a,a^\dagger]}_{=\mathbb{1}}]=0,\ \forall x$$

$$(7)\quad = \underbrace{a^\dagger e^{-\bar{a}a}}_{=:(7a)} - \bar{a}\cdot e^{-\bar{a}a}$$

- Similarly,

$$a e^{a a^\dagger} = e^{a a^\dagger}\left(e^{-a a^\dagger} a e^{a a^\dagger}\right) = e^{a a^\dagger}\left\{a + \overbrace{[a^\dagger,a]}^{=-\mathbb{1}}(-a) + 0\right\}$$

$$(8)\quad = \underbrace{e^{a a^\dagger} a}_{=(7b)} + a e^{a a^\dagger}$$

- Substituting these into (2) and using cyclicity of trace yield (7a) and (7b) cancel term (2a) and thus we get:

$$\partial_t \eta_I = -\frac{\gamma}{2}\left\{\bar{a}\cdot\mathrm{Tr}\left(\rho_I e^{a a^\dagger}(-a)e^{-\bar{a}a}\right) + a\cdot\mathrm{Tr}\left(\rho_I a^\dagger e^{a a^\dagger}e^{-\bar{a}a}\right)\right\}$$

$$(9)\quad \overset{(3)\&(4)}{=} -\frac{\gamma}{2}\left\{\bar{a}\frac{\partial}{\partial\bar{a}}\eta_I + a\frac{\partial}{\partial a}\eta_I\right\}$$

- Now, $\frac{\partial}{\partial(\ln a)} = \frac{\partial a}{\partial(\ln a)}\frac{\partial}{\partial a} = \left(\frac{\partial \ln a}{\partial a}\right)^{-1}\frac{\partial}{\partial a} = a\frac{\partial}{\partial a}$ and $\frac{\partial}{\partial(\ln\bar{a})} = \bar{a}\frac{\partial}{\partial\bar{a}}$

- Thus (9) is equivalent to the PDE:

$$(10)\quad \left[\frac{\partial}{\partial t} + \frac{\gamma}{2}\left(\frac{\partial}{\partial(\ln a)} + \frac{\partial}{\partial(\ln\bar{a})}\right)\right]\eta_I = 0$$

8.3 b) Method of characteristics is an overkill for our problem. Anyway, MC for a general linear PDE:

$$Lu(x) \equiv b(x)\cdot Du + c(x)u(x) = 0 \quad , \; x \equiv (x^1,\dots,x^d) \in \mathbb{R}^d, \; D = \left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^d}\right)$$

yields (Details: L.C. Evans, Partial diff. Equations AMS, p. 99-100)

$$(1) \quad \begin{cases} \frac{\partial}{\partial s} x(s) = b(x(s)) \\ \frac{\partial}{\partial s} z(s) = -c(x(s))\, z(s) \end{cases}$$

where s is a technical, TIME LIKE, parameter that parametrizes a curve $s \mapsto (x(s), z(s))$ on the solution manifold $\{(x, u(x)) : Lu(x)=0,\ x\in K\}$ of the PDE. Here $K \subset \mathbb{R}^d$ is the domain where PDE is to be solved. By solving (1) one hopes to be able to relate the general value of $u(x)$ to a know boundary value of $u(x_0) \in \partial K$ by connecting

$$(x(0), z(0)) = (x_0, u(x_0))$$
$$(x(s_1), z(s_1)) = (x, u(x)).$$

In our case,

$$x = (x^1,x^2,x^3) = (t, a, \bar{a})$$
$$b(x) = \left(1, \frac{\gamma}{2}x^1, \frac{\gamma}{2}x^2\right)$$
$$c(x) = 0$$
$$K = \{(x^1,x^2,x^3)\in\mathbb{R}^3 : x^1 \geq 0\}$$
$$\Rightarrow \partial K = \{0\}\times\mathbb{R}^2$$

NOTE: We could also chose $x = (t, \ln a, \ln\bar{a})$ etc. and obtain the same result.

Also note: We consider initially $\mathbb{R}^d$ but everything goes smoothly in $\mathbb{C}^d$ as well...

- Of course, we want to r
- Let's choose parameter s such that: $x^1(s) = s$.
- Then (1) yields:

$$\left.\begin{aligned} \dot{a}(s) &= \frac{\gamma}{2} a(s) \\ \dot{\bar{a}}(s) &= \frac{\gamma}{2}\bar{a}(s) \\ \dot{z}(s) &= 0 \end{aligned}\right\} \Rightarrow \begin{cases} a(s) = a_0\cdot e^{\frac{\gamma}{2}s} \\ \bar{a}(s) = \bar{a}_0\, e^{\frac{\gamma}{2}s} \\ z(s) = z_0 \end{cases} \quad \text{for all } a_0, \bar{a}_0, z_0 \in \mathbb{R}.$$

- Let $\chi_I(0, a, \bar{a}) =: \chi(a,\bar{a}) \equiv \mathrm{Tr}\left(\rho(0)\, e^{a a^\dagger} e^{-\bar{a} a}\right)$.
- Then:

$$\chi(a_0, \bar{a}_0) = z(0) = z(s) = \chi_I(x(s)) = \chi_I\left(s, a_0 e^{\frac{\gamma}{2}s}, \bar{a}_0 e^{\frac{\gamma}{2}s}\right)$$

- No setting $a = a_0\cdot e^{\frac{\gamma}{2}s}$, $\bar{a} = \bar{a}_0\cdot e^{\frac{\gamma}{2}s}$ and $t = s$:

$$\chi_I(t, a, \bar{a}) = \chi\left(a e^{-\frac{\gamma}{2}t}, \bar{a} e^{-\frac{\gamma}{2}t}\right)$$

- It's of course easy to check by substituting into PDE that this is solution for $a, \bar{a} \in \mathbb{C}$ as well.

8.3 c) Let $\chi_S(t, \alpha, \bar\alpha) = \mathrm{Tr}\left(\rho(t)\, e^{\alpha a^\dagger} e^{-\bar\alpha a}\right)$

- This solves PDE:

$$\left[\frac{\partial}{\partial t} + \left(\frac{\gamma}{2} - i\omega\right)\frac{\partial}{\partial \ln \alpha} + \left(\frac{\gamma}{2} + i\omega\right)\frac{\partial}{\partial \ln \bar\alpha}\right]\chi_S = 0$$

- To obtain this one works just like in a);
Indeed, there is a one extra term in (2) coming from
$$\mathrm{Tr}\left(-i[H_0, \rho]\, e^{\alpha a^\dagger} e^{-\bar\alpha a}\right) = -i\omega \underbrace{\mathrm{Tr}\left(\rho\, e^{\alpha a^\dagger} e^{-\bar\alpha a} a^\dagger a\right)}_{=(17b)} + i\omega \underbrace{\mathrm{Tr}\left(\rho\, a^\dagger a\, e^{\cdots}\right)}_{=(17c)}$$

- Terms (17b) and (17c) are handled just like terms (2b) and (2c) respectively.

- Solution is obtained by using MC as before as well:

$$\chi_S(t, \alpha, \bar\alpha) = \chi\left(\alpha \cdot e^{-\left(\frac{\gamma}{2} - i\omega\right)t},\ \bar\alpha \cdot e^{-\left(\frac{\gamma}{2} + i\omega\right)t}\right)$$