7.1 - Let $\mathcal{D}(\mathcal{H}) := \{ A \in \mathcal{B}(\mathcal{H}) : A \geq 0 ,\ \mathrm{Tr}(A) = 1 \}$

- A super operator Φ maps $\mathcal{D}(\mathcal{H}) \circlearrowleft$
- Since Φ is linear we may write $\tilde{\rho} := \Phi(\rho)$, $\tilde{\rho}_{ij} = \langle i | \tilde{\rho} | j \rangle$, $\rho_{ij} = \langle i | \rho | j \rangle$
as: $\rho_{ij} = \rho^R_{ij} + i \rho^I_{ij}$, $\tilde{\rho}_{ij} = \tilde{\rho}^R_{ij} + i \tilde{\rho}^I_{ij}$, with $\rho^I_{ij}, \rho^R_{ij}, \ldots \in \mathbb{R}$.
- Since A^R_{ij} $(i \leq j)$ and A^I_{ij} $(i < j)$ specify $A^\dagger = A$
we can use linearity of Φ to write

$$\tilde{\rho}^R_{ij} = \sum_{k \leq l} \alpha^{RR}_{ij;kl} \rho^R_{kl} + \sum_{k<l} \alpha^{RI}_{ij;kl} \rho^I_{kl} \quad \text{for } i \leq j \qquad (1)$$

$$\tilde{\rho}^I_{ij} = \sum_{k \leq l} \alpha^{IR}_{ij;kl} \rho^R_{kl} + \sum_{k<l} \alpha^{II}_{ij;kl} \rho^I_{kl} \quad \text{for } i < j \qquad (2)$$

where $\alpha^{XY}_{ij;kl} \in \mathbb{R}$ for $X, Y \in \{R, I\}$.

- The number of independent coefficient $\alpha^{RX}_{ij;kl}$ given by (1) is thus:

$$\frac{M(M+1)}{2} \cdot \left\{ \frac{M(M+1)}{2} + \frac{M(M-1)}{2} \right\} = \frac{M(M+1)}{2} \cdot M^2$$

- The number of independent coefficient $\alpha^{I,X}_{ij;kl}$ given by (2) is

$$\frac{M(M-1)}{2} \cdot \left\{ \frac{M(M+1)}{2} + \frac{M(M-1)}{2} \right\} = \frac{M(M-1)}{2} M^2$$

So: $\Phi(\rho)^\dagger = \Phi(\rho)$ for $\rho^\dagger = \rho$ $\Rightarrow$ M^4 $\mathbb{R}$-parameters.

- Now, $\mathrm{Tr}(\Phi(\rho)) = \mathrm{Tr}(\rho)$ adds one extra constraint:

$\Rightarrow$ $\dim(\mathcal{H})^4 - 1$ $\mathbb{R}$-parameters needed to specify a general super operator Φ on $\mathcal{H}$.

7-2 c) TO GET ONLY TWO NONZERO KRAUSS-OPERATORS N_j: $j=1,2$.
WE TRACE E W.R.T. BASE $\{|\tilde{e}_0\rangle, |\tilde{e}_1\rangle, |\tilde{e}_2\rangle\}$
WHERE $|\tilde{e}_1\rangle := |v_1\rangle \equiv \sqrt{1-p}\,|e_0\rangle + \sqrt{p}\,|e_1\rangle$
AND $|\tilde{e}_2\rangle$ IS OBTAINED BY ORTOGONALIZING $|v_2\rangle$:

$$|\tilde{e}_2\rangle := \frac{|v_2\rangle - |\tilde{e}_1\rangle\langle\tilde{e}_1|v_2\rangle}{\| \quad - \quad \|}$$

AND $|\tilde{e}_0\rangle$ IS CHOSEN TO BE ORTOGONAL TO $|\tilde{e}_1\rangle$ AND $|\tilde{e}_2\rangle$.

- SO LET'S WRITE

$$\begin{aligned} |v_2\rangle &= \langle\tilde{e}_1|v_2\rangle|\tilde{e}_1\rangle + \langle\tilde{e}_2|v_2\rangle|\tilde{e}_2\rangle \\ &= \langle v_1|v_2\rangle|\tilde{e}_1\rangle + \sqrt{1-|\langle v_1|v_2\rangle|^2}\,|\tilde{e}_2\rangle \\ &= (1-p)\cdot|\tilde{e}_1\rangle + \sqrt{2p-p^2}\cdot|\tilde{e}_2\rangle \end{aligned}$$

- NOW, WE MAY CALCULATE:

$$\begin{aligned} N_1 &= (\mathbb{1}\otimes\langle\tilde{e}_1|)\,U\,(\mathbb{1}\otimes|e_0\rangle) = (\mathbb{1}\otimes\langle\tilde{e}_1|)\big(|1,v_1\rangle\langle 1,e_0| + |2,v_2\rangle\langle 2,e_0| + \ldots\big)(\mathbb{1}\otimes|e_0\rangle) \\ &= (\mathbb{1}\otimes\langle\tilde{e}_1|)\big(|1,v_1\rangle\langle 1| + |2,v_2\rangle\langle 2|\big) \\ &= (\mathbb{1}\otimes\langle\tilde{e}_1|)\big(|1,\tilde{e}_1\rangle\langle 1| + (1-p)|2,\tilde{e}_1\rangle\langle 2| + \sqrt{2p-p^2}\,|2,\tilde{e}_2\rangle\langle 2|\big) \\ &= |1\rangle\langle 1| + (1-p)|2\rangle\langle 2| \;\hat{=}\; \begin{bmatrix} 1 & 0 \\ 0 & 1-p \end{bmatrix} \end{aligned}$$

- AND SIMILARLY:

$$\begin{aligned} N_2 &= (\mathbb{1}\otimes\langle\tilde{e}_2|)\big(|1,\tilde{e}_1\rangle\langle 1| + (1-p)|2,\tilde{e}_1\rangle\langle 2| + \sqrt{2p-p^2}\,|2,\tilde{e}_2\rangle\langle 2|\big) \\ &= \sqrt{2p-p^2}\,|2\rangle\langle 2| \;\hat{=}\; \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2p-p^2} \end{bmatrix} \end{aligned}$$

7.2 - WE KNOW:

$$|k,e_0\rangle \overset{U}{\longmapsto} |k\rangle \otimes |v_k\rangle \qquad (k=1,2)$$

WITH $|v_2\rangle = \sqrt{1-p}\,|e_0\rangle + \sqrt{p}\cdot|e_k\rangle$.

(a) Kraus-operators M_α are obtained from the general expression:

THE relevant part of EXPANSION $U = \sum U_{ij,0\alpha} |i,e_j\rangle\langle k,e_\alpha|$

$$(*)\quad M_\alpha = (\mathbb{1}_S \otimes \langle e_\alpha|)\, U\, (\mathbb{1}\otimes|e_0\rangle)$$
$$= (\mathbb{1}_S \otimes \langle e_\alpha|)\left\{|1,v_1\rangle\langle 1,e_0| + |2,v_2\rangle\langle 2,e_0| + \dots\right\}(\mathbb{1}\otimes|e_0\rangle)$$

So:

$$M_0 = |1\rangle\langle e_0|v_1\rangle\langle 1| + |2\rangle\langle e_0|v_2\rangle\langle 2|$$
$$= \sqrt{1-p}\,|1\rangle\langle 1| + \sqrt{1-p}\,|2\rangle\langle 2|$$
$$= \sqrt{1-p}\,\mathbb{1}$$

$$M_1 = |1\rangle\langle e_1|v_1\rangle\langle 1| + |2\rangle\langle e_1|v_2\rangle\langle 2|$$
$$= \sqrt{p}\,|e_1\rangle\langle e_1| + 0$$

$$M_2 = \sqrt{p}\cdot|e_2\rangle\langle e_2| \quad \text{similarly.}$$

(b) Let $\mathcal{H}_S \cong \mathbb{C}^2$ and $|1\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|2\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

THEN $M_0 = \sqrt{1-p}\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $M_1 = \sqrt{p}\cdot\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $M_2 = \sqrt{p}\begin{bmatrix}0&0\\0&1\end{bmatrix}$

And thus
$$\Phi(\rho) = \sum_\alpha M_\alpha \begin{bmatrix}\rho_{11}&\rho_{12}\\\rho_{21}&\rho_{22}\end{bmatrix} M_\alpha^\dagger$$
$$= (1-p)\begin{bmatrix}\rho_{11}&\rho_{12}\\\rho_{21}&\rho_{22}\end{bmatrix} + p\begin{bmatrix}\rho_{11}&0\\0&0\end{bmatrix} + p\begin{bmatrix}0&0\\0&\rho_{22}\end{bmatrix}$$
$$= \begin{bmatrix}\rho_{11}&(1-p)\,\rho_{21}\\(1-p)\,\rho_{21}&\rho_{22}\end{bmatrix}$$

- by iteration:

$$\langle i|\Phi^M(\rho)|j\rangle = \delta_{ij}\langle i|\rho|j\rangle + (1-\delta_{ij})\,e^{-\frac{M}{\tau}}\langle i|\rho|j\rangle$$

where $-\frac{1}{\tau} = \ln(1-p)$, and $\tau > 0$ is called decoherence time scale

continues...

7.3 a) GENERALLY

IN OUR CASE $|E\rangle = |e_0\rangle$ ALWAYS.

$$M_\alpha \equiv (\mathbb{1} \otimes \langle e_\alpha|)\, U\, (\mathbb{1} \otimes |E\rangle)$$
$$= \Big(\mathbb{1} \otimes \Big\{\sum_\beta \langle e_\alpha|\tilde{e}_\beta\rangle \langle \tilde{e}_\beta|\Big\}\Big)\, U\, (\mathbb{1} \otimes |E\rangle)$$
$$= \sum_\beta \langle e_\alpha|\tilde{e}_\beta\rangle\, (\mathbb{1} \otimes \langle \tilde{e}_\beta|)\, U\, (\mathbb{1} \otimes |E\rangle)$$
$$\equiv \sum_\beta U_{\alpha\beta}\, \tilde{M}_\beta$$

- SO IN OUR CASE THE MATRIX $U \equiv [U_{\alpha\beta}]$ CONSISTS OF THE COLUMN VECTORS $|\tilde{e}_\beta\rangle$ IN CANONICAL BASE $|e_0\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}, |e_1\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}, |e_2\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$

- by ON-PROJECTING:

$$|\tilde{e}_1\rangle \equiv \begin{bmatrix}\sqrt{1-p}\\ \sqrt{p}\\ 0\end{bmatrix}, \quad |\tilde{e}_2\rangle = \frac{1}{\sqrt{2(1-p+p^2)}}\begin{bmatrix}p\\ -(1-p)\\ 1\end{bmatrix}$$

WHILE $|\tilde{e}_0\rangle$ IS OBTAINED BY NORMALIZING: $|v_1\rangle \times |v_2\rangle = \begin{bmatrix}p\\ -\sqrt{p(1-p)}\\ -\sqrt{p(1-p)}\end{bmatrix}$

$$|\tilde{e}_0\rangle = \frac{1}{\sqrt{2-p}}\begin{bmatrix}\sqrt{p}\\ -\sqrt{1-p}\\ -\sqrt{1-p}\end{bmatrix}$$

- SO

$$U = \begin{bmatrix} \sqrt{\frac{p}{2-p}} & \sqrt{1-p} & \frac{p}{\sqrt{2(1-p+p^2)}} \\ -\sqrt{\frac{1-p}{2-p}} & \sqrt{p} & \frac{-(1-p)}{\sqrt{2(1-p+p^2)}} \\ -\sqrt{\frac{1-p}{2-p}} & 0 & \frac{1}{\sqrt{2(1-p+p^2)}} \end{bmatrix}$$

b) IF $\langle e_1|e_2\rangle = 1-\varepsilon$ THEN

$$\langle v_1|v_2\rangle = (1-p) + p\cdot(1-\varepsilon) =: 1-Q \in (0,1)$$

- WE MAY NOW REPEAT THE ANALYSIS IN 7.2 c), i.e., WE DEFINE $|\tilde{e}_1\rangle := |v_1\rangle$, $|\tilde{e}_2\rangle$ s.t. $sp\{|v_1\rangle, |v_2\rangle\} = sp\{|\tilde{e}_1\rangle, |\tilde{e}_2\rangle\}$ AND $\langle \tilde{e}_i|\tilde{e}_j\rangle = \delta_{ij}$

- SINCE $|v_1\rangle$ and $|v_2\rangle$ are UNIT VECTORS:

$$|v_2\rangle = (1-Q)\,|\tilde{e}_1\rangle + \sqrt{1-(1-Q)^2}\,|\tilde{e}_2\rangle$$

AND WE GET:

$$N_1 \cong \begin{bmatrix}1 & 0\\ 0 & 1-Q\end{bmatrix} \quad \text{AND} \quad N_2 = \begin{bmatrix}0 & 0\\ 0 & \sqrt{2Q-Q^2}\end{bmatrix}$$

as before.