6.1

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a) Set $\tilde{H}_l := -it H_l$:

$$U(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\tilde{H}_1 \otimes 1\!\!1 + 1\!\!1 \otimes \tilde{H}_2 \right)^n$$

- Now, $(A \otimes 1\!\!1)(1\!\!1 \otimes B) = A \otimes B = (1\!\!1 \otimes B)(A \otimes 1\!\!1)$ and thus:

$$U(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} H_1^{j} \otimes H_2^{n-j}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{j!(n-j)!} H_1^{j} \otimes H_2^{n-j}$$

- In the last double sum any $H_1^{n_1} \otimes H_2^{n_2}$, $n_1, n_2 \in \mathbb{N}_0$ appears exactly once and the coefficient is clearly $\frac{1}{n_1! n_2!}$.

Thus:

$$U(t) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{1}{n_1! n_2!} H_1^{n_1} \otimes H_2^{n_2}$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left(\frac{1}{n_1!} H_1^{n_1} \right) \otimes \left(\frac{1}{n_2!} H^{n_2} \right)$$

$$= \left(\sum_{n_1=0}^{\infty} \frac{1}{n_1!} H_1^{n_1} \right) \otimes \left(\sum_{n_2=0}^{\infty} \frac{1}{n_2!} H^{n_2} \right)$$

$$\equiv U_1(t) \otimes U_2(t) \quad \blacksquare$$

b) S-number of $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is the number $m \in \mathbb{N}$ in $|\psi\rangle = \sum_{i=1}^{m} \psi_i |\alpha_i, \beta_i\rangle$, with $\{|\alpha_i\rangle\}$ and $\{|\beta_j\rangle\}$ ON-bases for $\mathcal{H}_1$ and $\mathcal{H}_2$ resp., and $\psi_i \neq 0$.

- Now, $$U_1 \otimes U_2 |\psi\rangle = \sum_{i=1}^{m} \psi_i \underbrace{U_1 |\alpha_i\rangle}_{=: |\tilde{\alpha}_i\rangle} \otimes \underbrace{U_2 |\beta_i\rangle}_{=: |\tilde{\beta}_i\rangle}$$

$$= \sum_{i=1}^{m} \psi_i |\tilde{\alpha}_i, \tilde{\beta}_i\rangle \quad (*)$$

- Since U_l are unitary $\{|\tilde{\alpha}_i\rangle\}, \{|\tilde{\beta}_j\rangle\}$ are ON-bases as well.

$\Rightarrow$ $(*)$ is S-decomposition $\Rightarrow$ S-number is conserved.

- S-number of $\rho \in S_1(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is the maximum $m := \max\{m_j : j = 1, \dots, l\}$ where

$$\rho = \sum_{j=1}^{l} p_j |\psi_j\rangle\langle\psi_j| \quad , \quad p_j \neq 0 \quad \langle\psi_i|\psi_j\rangle = \delta_{ij}$$

and m_j is S-number of $|\psi_j\rangle$

- Clearly, $$U \rho U^\dagger = \sum_{j=1}^{l} p_j \underbrace{U |\psi_j\rangle\langle\psi_j| U^\dagger}_{= |\tilde{\psi}_j\rangle\langle\tilde{\psi}_j|}$$

$\Rightarrow$ $\langle\tilde{\psi}_i|\tilde{\psi}_j\rangle = \delta_{ij}$ and $|\tilde{\psi}_j\rangle$ and $|\psi_j\rangle$ have same S-number m_j.

$\Rightarrow$ S-number of ρ conserved by $U = U_1 \otimes U_2$ $\blacksquare$

(5.2) IT SUFFICES TO DO ONLY b), SINCE IT IMPLIES a)

- LET $\varphi \in \mathcal{H}_A \otimes \mathcal{H}_B$ BE THE STATE.
- LET $\rho_A = \sum_{i=1}^{m} p_i |\alpha_i\rangle\langle\alpha_i|$ FOR SOME ON-BASIS OF $\mathcal{H}_A$
- WE MAY THEN WRITE:

$$|\varphi\rangle = \sum_{ij} \varphi_{ij} |\alpha_i, \hat{\beta}_j\rangle, \text{ where } \{|\hat{\beta}_j\rangle\} \text{ IS SOME ON-BASIS OF } \mathcal{H}_B.$$

$$= \sum_i |\alpha_i\rangle \otimes \underbrace{\Big(\sum_j \varphi_{ij} |\hat{\beta}_j\rangle\Big)}_{=:|\tilde{\beta}_i\rangle} =: \sum_i |\alpha_i, \tilde{\beta}_i\rangle$$

- BY PROCEEDING LIKE IN THE LECTURE NOTES WE GET:

$$\sum_i p_i |\alpha_i\rangle\langle\alpha_i| = \rho_A = \mathrm{Tr}_B\Big[\sum_{ii'} |\alpha_i, \tilde{\beta}_i\rangle\langle\alpha_{i'}, \tilde{\beta}_{i'}|\Big]$$

$$= \sum_{ii'} |\alpha_i\rangle\langle\alpha_{i'}| \Big(\sum_k \langle k|\tilde{\beta}_i\rangle\langle\tilde{\beta}_{i'}|k\rangle\Big) \quad \{|k\rangle\} \text{ ON-BASIS FOR } \mathcal{H}_B$$

$$= \sum_{ii'} \langle\tilde{\beta}_{i'}|\tilde{\beta}_i\rangle |\alpha_i\rangle\langle\alpha_{i'}|$$

$\Rightarrow$ MUST HAVE $\langle\tilde{\beta}_{i'}|\tilde{\beta}_i\rangle = p_i \cdot \delta_{ii'} \Rightarrow |\beta_j\rangle := p_j^{-1/2} |\tilde{\beta}_j\rangle$ ON-BASIS.

$$\Rightarrow |\varphi\rangle = \sum_i \sqrt{p_i} |\alpha_i, \beta_i\rangle \Rightarrow \rho_B = \mathrm{Tr}_A\big[|\varphi\rangle\langle\varphi|\big]$$

$$= \sum_{i=1}^{m} p_i |\beta_i\rangle\langle\beta_i|$$

- SO: IF WE KNOW ρ_A AND ρ_B WE KNOW PROJECTORS $|\alpha_i\rangle\langle\alpha_i|$, $|\beta_j\rangle\langle\beta_j|$ AND THE NUMBERS p_j
- THIS IS NOT ENOUGH TO DETERMINE $|\varphi\rangle$ THOUGH.

INDEED, $|\tilde{\varphi}\rangle := \sum_{j=1}^{m} e^{i\theta_j}\sqrt{p_j} |\alpha_j, \beta_j\rangle$, $\theta_j \in \mathbb{R}$

YIELDS $\mathrm{Tr}_C(|\tilde{\varphi}\rangle\langle\tilde{\varphi}|) = \mathrm{Tr}_C(|\varphi\rangle\langle\varphi|)$, $C = A, B$

- MOREOVER, BESIDES THE GLOBAL PHASE θ IN $e^{i\theta}|\varphi\rangle$ WHICH HAVE NO PHYSICAL SIGNIFICANCE THE RELATIVE PHASES $\hat{\theta}_j := \theta_j - \theta_1$ $j = 1, \dots, m$ ALL MATTER.

- INDEED:

$$|\langle\varphi|P_j|\varphi\rangle|^2 = |\langle\varphi'|P_j|\varphi'\rangle|^2 \quad \forall j = 1, \dots, m$$

WHERE $P_j = |\phi_j\rangle\langle\phi_j|$, $|\phi_j\rangle = \frac{1}{\sqrt{2}}\big(|\alpha_1, \beta_1\rangle + |\alpha_j, \beta_j\rangle\big)$

IFF $\theta_j - \theta_1 = \theta_j' - \theta_1'$ $\forall j = 1, \dots, m$

- IN OTHER WORDS, BESIDES THE KNOWLEDGE OF ρ_A, ρ_B ONE MUST KNOW, $\hat{\theta}_j : j = 2, \dots, m$.

- IF ρ_A (AND THUS ALSO ρ_B) HAS NON-DEGENERATE SPECTRUM

6.2 CONTINUES...

THEN $\{p_j\}, \{|\alpha_j\rangle\langle\alpha_j|\}, \{|\beta_j\rangle\langle\beta_j|\}, \{\theta_j - \theta_1\}$ DETERMINE $|\psi\rangle$

- IF SPECTRUM IS DEGENERATE, E.G., SAY, $p_1 = p_2$

THEN EVEN MORE EXTRA KNOWLEDGE IS NEEDED:

$$|\psi\rangle = p_1 |\alpha_1, \beta_1\rangle + p_2 |\alpha_2, \beta_2\rangle + \dots$$

and

$$|\psi\rangle = p_1 |\alpha_1, \beta_2\rangle + p_2 |\alpha_2, \beta_1\rangle + \dots$$

both yield same ρ_A, ρ_B EVEN IF $\theta_j = \theta_j' \;, \forall j$.

- AN EXTREME CASE IS MAXIMALLY ENTANGLED VECTOR

$$|\psi\rangle = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} |\alpha_i, \beta_i\rangle \;, \quad \dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = M$$

$$\Rightarrow \rho_A = \rho_B = \frac{1}{M} \mathbb{1} \quad \forall \text{ ON-BASES } \{|\alpha_i\rangle\}, \{|\beta_i\rangle\}.$$

6.3 : $\mathcal{D} := \{ \rho \in \mathbb{C}^{M \times M} : \rho^\dagger = \rho ,\ \rho \geq 0 ,\ \mathrm{Tr}(\rho) = 1 \}$

- THE SET OF HERMITIAN MATRICES IS M^2 $\mathbb{R}$-dim

$\overline{\rho_{ji}} = (\rho^*)_{ij} = \rho_{ij}$ $\triangleright$ WRITE $\rho_{ij} = \rho^R_{ij} + i \rho^I_{ij}$, $\rho^R_{ij}, \rho^I_{ij} \in \mathbb{R}$

$\Rightarrow$ $\rho^I_{ii} = 0$, $\rho^R_{ii} \Rightarrow$ M - degrees of freedom.

$\rho_{ij} = \rho^R_{ij} + i \rho^I_{ij}$ for $i < j$ CAN BE CHOSEN freely

$\Rightarrow \frac{M(M-1)}{2} \cdot 2 = M^2 - M$ degrees of freedom.

$\Rightarrow$ THUS SELF-ADJOINTNESS GIVES $M^2 = (M^2 - M) + M$ $\mathbb{R}$-dim.

HOWEVER, SINCE $\mathrm{Tr}(\rho) = \sum_i \rho^R_{ii} = 1 \Rightarrow 1$ ρ^R_{ii} IS DETERMINED by others

$\dim_{\mathbb{R}}(\mathcal{D}) = M^2 - 1$ $\boxtimes$

6.4

(a) $|\psi\rangle = \frac{1}{\sqrt{2}} e_1 \otimes v_2 + \frac{1}{\sqrt{2}} e_2 \otimes v_2$

$v_1 = \frac{1}{2}(e_1 + \sqrt{3}\, e_2)$

$v_2 = \frac{1}{2}(\sqrt{3} e_1 + e_2)$

$\Rightarrow \|v_i\| = 1 \Rightarrow \begin{cases} |v_1\rangle\langle v_1| = \frac{1}{4}\begin{bmatrix} 1 \\ \sqrt{3}\end{bmatrix}\begin{bmatrix} 1 & \sqrt{3}\end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3\end{bmatrix} \\ |v_2\rangle\langle v_2| = \frac{1}{4}\begin{bmatrix} \sqrt{3} \\ 1\end{bmatrix}\begin{bmatrix} \sqrt{3} & 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1\end{bmatrix} \end{cases}$

Now; $\rho_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|)$ - USE basis $|i\rangle = e_i$ FOR A-TRACE.

$= \frac{1}{2}|v_1\rangle\langle v_1| + \frac{1}{2}|v_2\rangle\langle v_2| = \frac{1}{8}\begin{bmatrix} 4 & 2\sqrt{3} \\ 2\sqrt{3} & 4\end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1\end{bmatrix}$

TO CALCULATE ρ_A WE NOTE THAT:

$|\psi\rangle = \frac{1}{2\sqrt{2}}\{ |11\rangle + \sqrt{3}|12\rangle + \sqrt{3}|21\rangle + |22\rangle \}$

- THIS IS SYMMETRIC W.R.T. A AND B $\Rightarrow$ WE CAN WRITE

$|\psi\rangle = \frac{1}{\sqrt{2}} v_1 \otimes e_1 + \frac{1}{\sqrt{2}} v_2 \otimes e_2$

$\Rightarrow \rho_A = \rho_B = \frac{1}{2}\begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1\end{bmatrix}$

(b) WE MUST CALCULATE $|\varphi_\pm\rangle$, $P_\pm$ S.T. $\rho_A = \rho_B = P_+|\varphi_+\rangle\langle\varphi_+| + P_-|\varphi_-\rangle\langle\varphi_-|$

$P_\pm$: Let $\tilde{\rho} = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1\end{bmatrix}$, i.e., $\rho_A = \frac{1}{2}\tilde{\rho}$

$0 = \det(\tilde{\rho} - \lambda \mathbb{1}) = \begin{vmatrix} 1-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1-\lambda\end{vmatrix} = (1-\lambda)^2 - \frac{3}{4} = \frac{1}{4} - 2\lambda + \lambda^2$

$\Rightarrow \lambda_\pm = \frac{1}{2}\left(2 \pm \sqrt{4-1}\right) = 1 \pm \frac{\sqrt{3}}{2}$

$\Rightarrow P_\pm = \frac{1}{2}\left(1 \pm \frac{\sqrt{3}}{2}\right)$

- Now, if $|\varphi_\pm\rangle$ are KNOWN then by $\underline{6.2}$:

$|\psi\rangle = \sqrt{P_+}\,|\varphi_+, \varphi_+\rangle + \sqrt{P_-}\,|\varphi_-, \varphi_-\rangle$

- To find out $|\varphi_\pm\rangle$ WE WRITE $|\tilde{\varphi}_\pm\rangle = \begin{bmatrix} 1 \\ x_\pm \end{bmatrix}$ AND SOLVE

$\frac{1}{2}\begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1\end{bmatrix}\begin{bmatrix} 1 \\ x_\pm\end{bmatrix} = \rho_A|\tilde{\varphi}_\pm\rangle = P_\pm|\tilde{\varphi}_\pm\rangle = \frac{1}{2}\left(1 \pm \frac{\sqrt{3}}{2}\right)\begin{bmatrix} 1 \\ x_\pm\end{bmatrix}$

- first component gives $1 + \frac{\sqrt{3}}{2}x_\pm = 1 \pm \frac{\sqrt{3}}{2} \Rightarrow x_\pm = \pm 1$

i.e., $|\varphi_\pm\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \pm 1\end{bmatrix}$

6.5 $\dim(\mathcal{H}_A) < \infty$

(a) Let $\rho = \sum_{i=1}^{m} p_i |\alpha_i\rangle\langle\alpha_i|$, $\{|\alpha_i\rangle\}$ ON.

Let $\mathcal{H}_B$ s.t. $\dim(\mathcal{H}_B) \geq m$.

Let $\{|\beta_j\rangle\}$ ON set $j=1,\dots,m$.

Set: $|\psi\rangle = \sum_i p_i^{1/2} |\alpha_i, \beta_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

$\Rightarrow \|\psi\|^2 = \sum (p_i^{1/2})^2 = 1$ and

and $\mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \mathrm{Tr}_B\left(\sum_{i,j} \sqrt{p_i p_j}\, |\alpha_i, \beta_i\rangle\langle\alpha_j, \beta_j|\right)$

$$= \sum_k (\mathbb{1} \otimes \langle\beta_k|)(\dots)(\mathbb{1} \otimes |\beta_k\rangle)$$

$$= \sum_{i,j,k} \sqrt{p_i p_j}\, |\alpha_i\rangle \underbrace{\langle\beta_k|\beta_i\rangle}_{=\delta_{ki}} \langle\alpha_j| \underbrace{\langle\beta_j|\beta_k\rangle}_{=\delta_{jk}}$$

$$= \sum_k p_k |\alpha_k\rangle\langle\alpha_k| \equiv \rho_A$$

(b) Let $\psi_1, \psi_2 \in \mathcal{H}_A \otimes \mathcal{H}_B$ s.t. $\rho_A = \mathrm{Tr}_B(|\psi_k\rangle\langle\psi_k|)$, $k=1,2$

- We can write

$|\psi_k\rangle = \sum_{i=1}^{m} p_i^{1/2} |\alpha_i, \beta_i^{(k)}\rangle$, where $\{|\beta_i^{(k)}\rangle\}$ for $k=1,2$ is ON-basis for $\mathcal{H}_B$

$\Rightarrow |\beta_i^{(2)}\rangle = U|\beta_i^{(1)}\rangle$, $i=1,\dots,m$ where U is unitary since $\{|\beta_i^{(k)}\rangle\}$ are ON-bases.

$$\Rightarrow |\psi_2\rangle = \sum_{i=1}^{m} p_i^{1/2} |\alpha_i, \beta_i^{(2)}\rangle$$

$$= \sum_{i=1}^{m} p_i^{1/2} |\alpha_i\rangle \otimes U|\beta_i^{(1)}\rangle$$

$$= \sum_{i=1}^{m} p_i^{1/2} (\mathbb{1} \otimes U) |\alpha_i, \beta_i^{(1)}\rangle$$

$$= (\mathbb{1} \otimes U)\left(\sum_{i=1}^{2} p_i^{1/2} |\alpha_i, \beta_i^{(1)}\rangle\right)$$

$$\equiv (\mathbb{1} \otimes U)|\psi_1\rangle$$

(6.6) WE MUST SHOW THAT THERE EXISTS A FUNCTION $V: \mathcal{P}(\mathcal{H}) \to [0,1]$
s.t.
(i) $V(0)=0$
(ii) $V(\mathbb{1}-E) = 1-V(E)$, $\forall E \in \mathcal{P}(\mathcal{H})$

- WE MAY TAKE $\mathcal{H} = \mathbb{C}^2$
$\Rightarrow$ A NON-TRIVIAL PROJECTOR (NOT 0 OR $\mathbb{1}$) IS $|\psi\rangle\langle\psi| =: E$, FOR SOME $\psi \in \mathbb{C}^2$
$\Rightarrow$ CAN BE WRITTEN AS: $E \equiv E(\hat{m}) := \frac{1}{2}(\mathbb{1} + \hat{m}\cdot\bar{\sigma})$
where $\bar{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ are Pauli-spin MATRIES and $\hat{m} \in \mathbb{R}^3$, $\|\hat{m}\| = 1$
IS UNIQUE. WE HAVE $\hat{m}\cdot\bar{\sigma} = \hat{m}_x\sigma_x + \hat{m}_y\sigma_y + \hat{m}_z\sigma_z \in \mathbb{C}^{2\times 2}$

- IN $\mathbb{C}^2$ ONE HAS $\mathbb{1} = E(\hat{m}) + E(-\hat{m})$
$\Rightarrow$ (ii) TAKES THE FORM:

$$f(-\hat{m}) = 1 - f(\hat{m}) \quad (*)$$

where $f(\hat{m}) := V(E(\hat{m}))$ defines a
function $f: \{x \in \mathbb{R}^3 : \|x\| = 1\} \to [0,1]$

- Moreover, IN LIGHT OF $(*)$ f CAN BE CHOSEN
completely freely ON $\{x \in \mathbb{R}^3 : \|x\| = 1, x_3 \geq 0\} =: D_+$
$\Rightarrow$ THE DIMENSION EVEN CONTINUOUS FUNCTIONS $C(D_+; [0,1])$
IS ∞.

- ON THE OTHER hand, the SPACE of DENSITY MATRICES
ON $\mathbb{C}^2$ IS FINITE.
$\Rightarrow$ NOT all V satisfying (i-ii) are of the form
$V(E) = \mathrm{Tr}(\rho E)$, $\rho \in$ "DENSITY MATRIX" ⊠