Quantum Probability: Exercise set 6 (17.3.2010 - Corrected version)

1. Let $k = 1, 2$. Suppose $\mathcal{H}_k$ are Hilbert spaces and $H_k \in \mathcal{B}(\mathcal{H}_k)$ are self-adjoint operators. Define $H := H_1 \otimes I_2 + I_1 \otimes H_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and set $U_k(t) := \mathrm{e}^{-\mathrm{i}tH_k}$ and $U(t) := \mathrm{e}^{-\mathrm{i}tH}$.

   (a) Prove
   $$U(t) \;=\; U_1(t) \otimes U_2(t) \qquad \text{for all} \qquad t \in \mathbb{R}\,.$$

   (b) How does the Schmidt number of a density operator ρ evolve under the Hamiltonian H?

2. (a) Show that $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is not determined by $\rho_1 = \mathrm{Tr}_2(|\psi\rangle\langle\psi|)$ and $\rho_2 = \mathrm{Tr}_1(|\psi\rangle\langle\psi|)$.

   (b) Suppose, that the non-zero eigenvalues of ρ_1 are non-degenerate. By writing the Schmidt-decomposition of ψ in a suitable basis argue that to specify ψ one must generally know $m-1$ real numbers $\{s_j : j = 1, \dots, m\}$ in addition to ρ_1, ρ_2 to determine ψ up to a global phase.

3. Suppose $\mathcal{H}$ is n-dimensional. Show that the real-dimension of the set of density matrices is $n^2 - 1$.

4. Let $\{e_1, e_2\}$ be a ortonormal basis of $\mathcal{H} = \mathbb{C}^2$. For the $\mathcal{H} \otimes \mathcal{H}$-state
   $$\psi \;=\; \frac{1}{\sqrt{2}} e_1 \otimes \left(\frac{1}{2} e_1 + \frac{\sqrt{3}}{2} e_2 \right) \;+\; \frac{1}{\sqrt{2}} e_2 \otimes \left(\frac{\sqrt{3}}{2} e_1 + \frac{1}{2} e_2 \right)$$

   (a) Compute $\rho_A = \mathrm{Tr}_2(|\psi\rangle\langle\psi|)$ and $\rho_B = \mathrm{Tr}_1(|\psi\rangle\langle\psi|)$;

   (b) Find the Schmidt decomposition of ψ.

5. Let $\mathcal{H}_A$ be finite dimensional and suppose ρ_A is a density matrix on $\mathcal{H}_A$.

   (a) Find another Hilbert space, $\mathcal{H}_B$, and a state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$.

   (b) Suppose $\psi_1, \psi_2 \in \mathcal{H}_A \otimes \mathcal{H}_B$ satisfy $\rho_A = \mathrm{Tr}_B(|\psi_k\rangle\langle\psi_k|)$, $k = 1, 2$. Prove that there exists an unitary operator U_B on $\mathcal{H}_B$ such that $\psi_2 = (I \otimes U_B)\psi_1$.

6. By finding a counterexample show that Gleason's theorem does not hold when $\dim(\mathcal{H}) = 2$.