Quantum Probability: Exercise set (4+5) (3.3.2010)

Unless otherwise stated, $\mathcal{H}$ denotes a complex separable Hilbert space with an innerproduct (x, y) and a norm $\|x\| = \sqrt{(x, x)}$. We denote by $\mathcal{F}_\pm$ the bosonic $(+)$ and fermionic $(-)$ Fock spaces. We write $[A, B] := AB - BA$ and $\{A, B\} := AB + BA$ for operators A, B. Denote by $\mathcal{L}(X)$ the set of linear operators $A : X \to X$.

1. (a) Prove that $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in \mathcal{H}$.

   (b) Express the inner product (x, y) for general $x, y \in \mathcal{H}$ in terms the norms $\|z\| = (z, z)$ where $z \in \mathcal{H}$ is a function of x, y. This shows that the inner product is completely determined by the norm $\| \bullet \|$.

2. (a) Show that for any $a(x) : P_\pm \mathcal{H}^n \to P_\pm \mathcal{H}^{n-1}$, $x \in \mathcal{H}$ and $n \in \mathbb{N}$.

   (b) Set $a^*_\pm(x) := P_\pm a^*(x)$. Show that $a^*_\pm(x) : P_\pm \mathcal{H}^n \to P_\pm \mathcal{H}^{n+1}$

3. Show that $\{a_-(x), a^*_-(y)\} = (x, y)I$.

4. Denote $\|A\|_{X \to Y} := \sup \{\|Ax\|_Y : x \in X, \|x\|_X \leq 1\}$. Show that:

   (a) $\|a(x)\|_{\mathcal{H}^n \to \mathcal{H}^{n-1}} \leq \sqrt{n}\, \|x\|$.

   (b) $\|a^*(x)\|_{\mathcal{H}^n \to \mathcal{H}^{n+1}} \leq \sqrt{n+1}\, \|x\|$.

5. Let A, B be bounded linear operators on $\mathcal{H}$. Show that
$$\mathrm{e}^{tA} B \mathrm{e}^{-tA} = B + [A, B] + \frac{[A, [A, B]]}{2!} t^2 + \frac{[A, [A, [A, B]]]}{3!} t^3 + \ldots$$

6. Using the previous exercise prove that $[A, [A, B]] = [B, [A, B]] = 0$ implies:
$$\mathrm{e}^{A+B} \;=\; \mathrm{e}^A \, \mathrm{e}^B \, \mathrm{e}^{-\frac{1}{2}[A,B]} \,.$$

7. Tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be considered as the set of equivalence classes in $\mathcal{H} := \mathcal{H}_1 \times \mathcal{H}_2$ such that $x \otimes y := \{(z, w) \in \mathcal{H} : L(z, w) = L(x, y)$ for all bilinear forms $L : \mathcal{H} \to \mathbb{C}\}$. If $A_k \in \mathcal{L}(\mathcal{H}_k)$, $k = 1, 2$, then $A_1 \otimes A_2 \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is defined by $(A_1 \otimes A_2)(x_1 \otimes x_2) = A_1 x_1 \otimes A_2 x_2$. Show that:

   (a) Tensor product of two self adjoint operators is self adjoint;

   (b) Tensor product of two unitary operators is unitary;

   (c) Tensor product of two projection operators is projection.

8. Let $\mathcal{H}_k$, $k = 1, 2$, be finite dimensional. Let ρ be a state (density matrix) on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Let M_1 be a self-adjoint operator on $\mathcal{H}_1$ so that $M := M_1 \otimes I$ is a self-adjoint operator on $\mathcal{H}$. Define the partial trace $\mathrm{tr}_2 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_1)$ by first setting $\rho_2(\alpha_1 \otimes \alpha_2) := \mathrm{tr}(\alpha_2) \cdot \alpha_1$ for one-dimensional projectors α_k, $k = 1, 2$, and then extending to general $\mathcal{L}(\mathcal{H})$ by linearity.

   (a) Show that $\mathrm{tr}\,(M\rho) = \mathrm{tr}\,(M_1 \rho_1)$.

   (b) Show that $(A, B) := \mathrm{tr}\,(A^* B)$ defines an inner product on $\mathcal{L}(\mathcal{H}_1)$ and thus makes $\mathcal{L}(\mathcal{H}_1)$ itself a Hilbert space. Find ortonormal basis of $\mathcal{L}(\mathcal{H}_1)$ w.r.t. this inner product.

   (c) Argue that ρ_1 is a unique choice which makes (a) hold for any measurement M of the form $M = M_1 \otimes I$. *Hint:* Consider a general mapping $f : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_1)$ that sends the states in $\mathcal{H}$ into the states in $\mathcal{H}_1$. Expand $f(\rho)$ by using the inner product of (b).

   - What is the physical meaning of this exercise?