Quantum Probability: Exercise set 3 (17.2.2010) – corrected version

This time it suffices to do only 4 exercises to get full points.

PMJP := "Pure Markov Jump Process";
Let $A \in \mathbb{C}^{n\times p}$. Define $A^\dagger \in \mathbb{C}^{p\times n}$ by $A^\dagger_{ij} = A^*_{ji}$. Denote $\langle x, y\rangle = \sum_{i=1}^d x_i^* y_i$, for $x, y \in \mathbb{C}^d$.

1. (a) Is $\tilde\tau : \Omega \to [0,\infty)$ defined by $t + \tilde\tau = \inf\{u \geq t : X_u \neq X_0\}$ a stopping time w.r.t. filtration generated by a cadlag process (X_t)?

   (b) Suppose X pure jump process. Define $\Delta_t := \max\{u \leq t : \lim_{s\nearrow u} X_s \neq X_u\}$ and assume that $Y_t := (X_t, \Delta_t)$ is Markov. Set $\tau := \inf\{t \geq 0 : X_t \neq X_0\}$. Can one always construct a function λ such that $\mathsf{P}(\tau \geq t | Y_0 = y) = \mathrm{e}^{-\lambda(y)t}$ for every $t \geq 0$? Prove this or provide a counter example.

2. (a) Denote $\dot u = \frac{\mathrm{d}u}{\mathrm{d}t}$. Find the solution of the one-dimensional ordinary differential equation $\dot x(t) = -\alpha x(t) + \dot f(t)$ with $x(0) = x$, where $\alpha > 0$ is a constant and f is some differentiable function on $\mathbb{R}$.

   (b) Let $A \in \mathbb{R}^{d\times d}$, $F \in \mathbb{R}^{d\times n}$ be constant matrices, and let (B_t) be a n-dimensional Brownian motion. Show that $\mathrm{d}X_t = -AX_t + F\mathrm{d}B_t$, $X_0 = x \in \mathbb{R}^d$ is solved by

   $$X_t = \mathrm{e}^{-tA}x + \int_0^t \mathrm{e}^{-(t-s)A} F \mathrm{d}B_s\,.$$

   Hint: Argue that (X_t) is Gaussian and consider 2-point correlations. You may also assume that $d = 1$ if you wish to avoid clutter.

   (c) What happens in (b) if A is positive definite, i.e., $x^{\mathrm{T}}Ax > 0$ for every $x \neq 0$.

3. Let X be a Gaussian on $\mathbb{C}^d$ such that $C = \mathsf{E}(XX^\dagger)$ is positive matrix. Calculate $W(\lambda) := \mathsf{E}\mathrm{e}^{\mathrm{i}\langle\lambda, X\rangle}$.

4. Let (X_t^1) and (X_t^2) be independent PMJP:s on $\mathbb{R}$ with jump rates μ^1 and μ^2, respectively. Define (X_t) on $\mathbb{R}^2$ and (Y_t) on $\mathbb{R}$ by setting $X_t = (X_t^1, X_t^2)$ and $Y_t := X_t^1 + X_t^2$, respectively. Let P_t and Q_t be the Markov transition kernels of X and Y, respectively.
   (a) Are X and Y are also PMJP:s?
   (b) Express transition probabilities $P_t(x, \{x\})$ and $P_t(x, B)$, $x \notin B$ in the first order of t as $t \to 0$.
   (c) Suppose $\mu^k(x, \mathrm{d}y) = f_k(y - x)\mathrm{d}y$ for $k = 1, 2$. Repeat (b) with P_t replaced by Q_t.

5. Let (X_t) be PMJP on $\mathbb{Z}$, such that $p(n,m) = p\delta_{m,n+1} + (1-p-q)\delta_{n,m} + q\delta_{m,n-1}$, where $p, q > 0$ and $\lambda(x) = \lambda > 0$ are constants and $p + q \leq 1$. (a) Find the generator A of X. (b) Calculate the semigroup e^{tA} of X: $(\mathrm{e}^{tA}u)(x) = \mathsf{E}_x u(X_t)$.