(2.1) **Kolmogorov's continuity theorem:**

- Suppose process $(X_t)_{t\geq 0}$ satisfies the condition:
  $\forall T>0,\ \exists \alpha,\beta,D\in(0,\infty)$ s.t. $\mathbb{E}(|X_t-X_s|^\alpha)\leq D\cdot|t-s|^{1+\beta}\quad \forall s,t\in[0,T]$
- Then there exists a continuous version of $(X_t)_{t\geq 0}$.

NOTE: Two processes (X_t) and (Y_t) are versions of each others if $\mathbb{P}[X_t=Y_t]=1\quad \forall t\in\mathbb{R}$

EXAMPLE: Take $\Omega:=[0,1]$, $\mathcal{F}:=$"Lebesgue measurable sets", $\mathbb{P}=$"Lebesgue measure on $[0,1]$"
Define two processes $X\equiv(X_t)_{t\in[0,1]}$, $Y\equiv(Y_t)_{t\in[0,1]}$ on $(\Omega,\mathcal{F},\mathbb{P})$ by setting:

$$X_t(\omega)=0,\ \forall t\in[0,1]$$
$$Y_t(\omega)=\begin{cases}1, & t=\omega\\ 0, & t\in[0,1]\setminus\{\omega\}\end{cases}$$

Then $\mathbb{P}[X_t=Y_t]=1$, i.e., X and Y are versions of each others. - finite dim. distr. all agree as well!

However, $\forall\omega\in\Omega$, $Y\equiv(t\mapsto Y_t(\omega))$ is discontinuous while $\forall\omega\in\Omega$, $X\equiv(t\mapsto X_t(\omega))$ is continuous. so $\mathbb{P}[Y \text{ continuous}]=0$, $\mathbb{P}[X\text{ continuous}]=1$

The actual exercise starts here:

- Let's show that for B.M. Kolmogorov's cont. theorem holds when $\alpha=4$, $\beta=1$.
- Since BM. is stationary it suffices to consider $s=0$:
- Because, $\mathbb{E}[e^{\lambda X}]=e^{\frac{\sigma^2}{2}\lambda^2}$, for a Gaussian $X\sim\mathcal{N}(0,\sigma^2)$ we have

$$\mathbb{E}[e^{\lambda B_t}]=e^{+\frac{t}{2}\lambda^2}\quad\text{and thus}\quad \mathbb{E}[|B_t-B_0|^4]=\mathbb{E}[B_t^4]=\frac{d^4}{d\lambda^4}\Big|_{\lambda=0}e^{\frac{t}{2}\lambda^2}$$

- Let $f(\lambda):=e^{\frac{t}{2}\lambda^2}$ so that

$$\frac{d^4}{d\lambda^4}\Big|_{\lambda=0}f(\lambda)=\frac{d^3}{d\lambda^3}\Big|_{\lambda=0}\{t\lambda\cdot f(\lambda)\}=\frac{d^2}{d\lambda^2}\Big|_{\lambda=0}\{(t+(t\lambda)^2)\cdot f(\lambda)\}$$
$$=\frac{d}{d\lambda}\Big|_{\lambda=0}\{(t\cdot(t\lambda)+(t\lambda)^3+2(t\lambda)t)\cdot f(\lambda)\}$$
$$=\Big\{\frac{d}{d\lambda}\Big|_{\lambda=0}(3t^2\lambda+\mathcal{O}(\lambda^2))\Big\}\cdot f(\lambda)=3t^2\Rightarrow \underline{D=3}\quad\boxtimes$$

(2.2) The transition density $p_{s,t}(x,y)$ of BM is specified by

$$\mathbb{E}\left[u(B_t)\,|\,B_s=x\right] = \int_{\mathbb{R}} p_{s,t}(x,y)\,u(y)\,dy$$

and is a simple normal distribution:

$$p_{s,t}(x,y) = \frac{1}{\sqrt{2\pi(t-s)}}\exp\left[-\frac{1}{2}\frac{(y-x)^2}{t-s}\right]$$

$$=: \rho_{t-s}(y-x)$$

$$\Rightarrow (*) \quad a\cdot\rho_{a^2\tau}(az) = \rho_\tau(z), \ \forall a>0, \forall z\in\mathbb{R}$$

- a continuous process (we always chose a continuous version if possible) is specified by finite dimensional distributions since values of a process for $\forall t\in\mathbb{Q} \equiv$ "rational numbers" determines the values for $\forall t\in\mathbb{R}$ by continuity, and $\{X_t=Y_t: t\in\mathbb{Q}\} = \bigcap_{j=1}^{\infty}\{X_{t_j}=Y_{t_j}\} = \lim_{N\to\infty}\bigcap_{j=1}^{N}\{X_{t_j}=Y_{t_j}\}$ where $\{t_j: j=1,2..\}=\mathbb{Q}$.
- Moreover, any open set in $\mathbb{R}$ can be expressed in terms of countable unions & intersections of the sets of form $\{x\in\mathbb{R}: x\le b\}$

$\Rightarrow$ we are done if we show:

$$\mathbb{P}\left[X_{t_j}\le x_j,\ j=1..N\right] = \mathbb{P}\left[B_{t_j}\le x_j,\ j=1...N\right]$$

$\forall$ finite sets $\{t_j\}$ and $\{x_j\}$.

- So: $\mathbb{P}\left[X_{t_j}\le x_j, \forall j\right] = \mathbb{P}\left[a^{-1}B_{a^2t_j}\le x_j, \forall j\right] = \mathbb{P}\left[B_{a^2t_j}\le a x_j, \forall j\right]$

$$= \int_{-\infty}^{ax_1}\int_{-\infty}^{ax_2}\cdots\int_{-\infty}^{ax_N}\prod_{j=1}^{N}\rho_{a^2(t_j-t_{j-1})}(y_j-y_{j-1})\,dy_j, \quad \text{with } t_0=y_0=0$$

- Let's make a change-of-variables: $y_j\mapsto z_j$ where $y_j=az_j$

$$\Rightarrow \mathbb{P}\left[X_{t_j}\le x_j, \forall j\right] = \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_N}\prod_{j=1}^{N}\underbrace{a\rho_{a^2(t_j-t_{j-1})}\left(a(z_j-z_{j-1})\right)}_{=\rho_{t_j-t_{j-1}}(z_j-z_{j-1})\ \text{- by } (*)}dz_j$$

$$= \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_N}\prod_{j=1}^{N}\rho_{t_j-t_{j-1}}(z_j-z_{j-1})\,dz_j$$

$$= \mathbb{P}\left[B_{t_j}\le x_j, \forall j\right]$$ - this completes part (a).

(b) - suppose $a^2s\ge t$:

$$\mathbb{E}\left[X_sB_t\right] = \mathbb{E}\left[\left(\tfrac{1}{a}B_{a^2s}\right)B_t\right] = \frac{1}{a}\mathbb{E}\left[\{(B_{a^2s}-B_t)+B_t\}B_t\right]$$

$$= \frac{1}{a}\mathbb{E}\left[(B_{a^2s}-B_t)B_t\right] + \frac{1}{a}\mathbb{E}\left[B_t^2\right]$$

$$= \frac{1}{a}\mathbb{E}\left\{\mathbb{E}\left[B_t\cdot(B_{a^2s}-B_t)\,|\,\mathcal{F}_t\right]\right\} + \frac{1}{a}t$$

$$= \frac{1}{a}\mathbb{E}\left\{B_t\cdot\underbrace{\mathbb{E}\left[B_{a^2s}-B_t\,|\,\mathcal{F}_t\right]}_{=\mathbb{E}\left[B_{a^2s}-B_t\right]=0}\right\} + \frac{1}{a}t$$

$= \frac{t}{a}$ - similarly, for $a^2s\le t$: $B_t=(B_t-B_{a^2s})+B_{a^2s}\cdots$ yields: $\mathbb{E}\left[X_sB_t\right] = as$

NOTE: This solution was kindly provided and typed by Mikko Pakkanen. There are some details (namely the convergence results which so that one really can ignore some small terms in the limit) which you can simply ignore.

In what follows, [RY] refers to REVUZ AND YOR: *Continuous Martingales and Brownian Motion*, 3rd edition.

1. Let $(\pi_n := \{0 = t_1^{(n)} < t_1^{(n)} < \cdots < t_{n+1}^{(n)} = t\})_{n=1}^{\infty}$ be a sequence of partitions of $[0, t]$, such that

$$\mathrm{mesh}(\pi_n) := \max_{0<k\leq n} |t_{k+1}^{(n)} - t_k^{(n)}| \xrightarrow[n\to\infty]{} 0.$$

The discrete integration by parts formula yields (since $B_0^3 = 0$)

$$\sum_{k=1}^{n} B_{t_k^{(n)}}^2 (B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}}) = B_t^3 - \sum_{k=1}^{n} B_{t_k^{(n)}} (B_{t_{k+1}^{(n)}}^2 - B_{t_k^{(n)}}^2) - \sum_{k=1}^{n} (B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}})(B_{t_{k+1}^{(n)}}^2 - B_{t_k^{(n)}}^2).$$

Moreover, using the identity

$$B_{t_{k+1}^{(n)}}^2 - B_{t_k^{(n)}}^2 = 2B_{t_k^{(n)}} (B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}}) + (B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}})^2,$$

we obtain

$$\sum_{k=1}^{n} B_{t_k^{(n)}}^2 (B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}}) = \frac{1}{3} B_t^3 - \sum_{k=1}^{n} B_{t_k^{(n)}} (B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}})^2 - \frac{1}{3} \sum_{k=1}^{n} (B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}})^3. \tag{1}$$

For the third term in the right hand side of (1), we have the bound

$$\left| \sum_{k=1}^{n} (B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}})^3 \right| \leq \max_{0<m\leq n} |B_{t_{m+1}^{(n)}} - B_{t_m^{(n)}}| \sum_{k=1}^{n} (B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}})^2,$$

where

$$\sum_{k=1}^{n} (B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}})^2 \xrightarrow[n\to\infty]{\mathbf{P}} t,$$

and since the paths $s \mapsto B_s(\omega)$, $s \in [0, t]$ are *uniformly continuous* for almost all $\omega \in \Omega$,

$$\max_{0<m\leq n} |B_{t_{m+1}^{(n)}} - B_{t_m^{(n)}}| \xrightarrow[n\to\infty]{\text{a.s.}} 0.$$

Since almost sure convergence implies convergence in probability, we have

$$\sum_{k=1}^{n} (B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}})^3 \xrightarrow[n\to\infty]{\mathbf{P}} 0.$$

Next we study the second term in the right hand side of (1). For any $M > 0$, let us denote

$$A_M := \left\{ \sup_{0\leq s\leq t} |B_s| \leq M \right\}.$$

Now, we have

$$\begin{aligned} &\mathbf{E}\left[\mathbf{1}_{A_M} \left\{ \sum_{k=1}^{n} B_{t_k^{(n)}} \left((B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}})^2 - (t_{k+1}^{(n)} - t_k^{(n)}) \right) \right\}^2 \right] \\ &\leq M^2 \sum_{k=1}^{n} \mathbf{E}\left[\left((B_{t_{k+1}^{(n)}} - B_{t_k^{(n)}})^2 - (t_{k+1}^{(n)} - t_k^{(n)}) \right)^2 \right], \end{aligned}$$

where

$$\mathbf{E}\left[\left((B_{t^{(n)}_{k+1}} - B_{t^{(n)}_k})^2 - (t^{(n)}_{k+1} - t^{(n)}_k)\right)^2\right] = \mathbf{E}\left[(B_{t^{(n)}_{k+1}} - B_{t^{(n)}_k})^4\right] + (t^{(n)}_{k+1} - t^{(n)}_k)^2$$
$$- 2\mathbf{E}\left[(B_{t^{(n)}_{k+1}} - B_{t^{(n)}_k})^2\right](t^{(n)}_{k+1} - t^{(n)}_k)$$
$$= 2(t^{(n)}_{k+1} - t^{(n)}_k)^2.$$

Hence,

$$\mathbf{E}\left[\mathbf{1}_{A_M}\left\{\sum_{k=1}^n B_{t^{(n)}_k}\left((B_{t^{(n)}_{k+1}} - B_{t^{(n)}_k})^2 - (t^{(n)}_{k+1} - t^{(n)}_k)\right)\right\}^2\right] \leq 2M^2 \sum_{k=1}^n (t^{(n)}_{k+1} - t^{(n)}_k)^2,$$

which converges to zero as $n \to \infty$, since $t \mapsto t$ is of finite variation, and thus, has zero quadratic variation. Now, let $\varepsilon > 0$ and $\delta > 0$. For brevity, denote

$$S_n := \sum_{k=1}^n B_{t^{(n)}_k}(B_{t^{(n)}_{k+1}} - B_{t^{(n)}_k})^2 \quad \text{and} \quad S'_n := \sum_{k=1}^n B_{t^{(n)}_k}(t^{(n)}_{k+1} - t^{(n)}_k).$$

Next, choose $M_{\delta,\varepsilon} > 0$, so that $\mathbf{P}(A^{\mathrm{c}}_{M_{\delta,\varepsilon}}) < \varepsilon/2$ and $n_{\delta,\varepsilon} \in \mathbb{N} \setminus \{0\}$, so that

$$\frac{2M^2_{\delta,\varepsilon}}{\delta^2} \sum_{k=1}^n (t^{(n)}_{k+1} - t^{(n)}_k)^2 < \frac{\varepsilon}{2}, \quad \text{for all } n \geq n_{\delta,\varepsilon}.$$

By Chebyshev's inequality, for any $n \geq n_{\delta,\varepsilon}$, we obtain

$$\mathbf{P}(|S_n - S'_n| > \delta) = \mathbf{P}(\{|S_n - S'_n| > \delta\} \cap A_{M_{\delta,\varepsilon}}) + \mathbf{P}(\{|S_n - S'_n| > \delta\} \cap A^{\mathrm{c}}_{M_{\delta,\varepsilon}})$$
$$\leq \frac{\mathbf{E}[\mathbf{1}_{A_{M_{\delta,\varepsilon}}}(S_n - S'_n)^2]}{\delta^2} + \mathbf{P}(A^{\mathrm{c}}_{M_{\delta,\varepsilon}}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

that is, $S_n - S'_n \xrightarrow[n\to\infty]{\mathbf{P}} 0$. Since S'_n is a Riemann sum, by almost sure continuity of paths,

$$S'_n \xrightarrow[n\to\infty]{\text{a.s.}} \int_0^t B_s \mathrm{d}s.$$

Hence,

$$\sum_{k=1}^n B^2_{t^{(n)}_k}(B_{t^{(n)}_{k+1}} - B_{t^{(n)}_k}) \xrightarrow[n\to\infty]{\mathbf{P}} \frac{1}{3}B^3_s - \int_0^t B_s \mathrm{d}s,$$

from which the assertion follows by Proposition IV.2.13 of [RY].

(2.3) b) TAKE $f(x) = \frac{1}{3}x^3$

THEN $f'(x) = x^2$, $f''(x) = 2x$

AND THUS
$$d\left(\frac{1}{3}B^3\right)_t = d(f \circ B)_t$$
$$= f'(B_t)\,dB_t + \frac{1}{2}f''(B_t)\,dt$$
$$= B_t^2\,dB_t + B_t\,dt$$

which is just a shorthand for:
$$\frac{1}{3}B_t^3 = \frac{1}{3}B_t^3 - \frac{1}{3}B_0^3 = \int_0^t d\left(\frac{1}{3}B^3\right)_s$$
$$= \int_0^t \left\{ B_s^2\,dB_s + B_s\,ds \right\}$$

or Equivalently:
$$\int_0^t B_s^2\,dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s\,ds$$

› For exercises 2.4 and 2.5 following result is very usefull:

[**Lévy characterization of B.M.:** Let $X \equiv (X_t)_{t\geq 0}$, $X_t = (X_t^1, \dots, X_t^d)$ be a process on $(\Omega, \mathcal{F}, \mathbb{P})$
"X is d-dim B.M." $\Longleftrightarrow$ "X and $X^i X^j - \delta^{ij} t$ are martingales for all i,j"]

▷ Recall: a process $M = (M_t)_{t\geq 0}$ a martingale w.r.t. increasing family $(\mathcal{F}_t)_{t\geq 0} \equiv \mathbb{F} \subset \mathcal{F}$ of σ-algebras iff $\mathbb{E}(M_t | \mathcal{F}_s) = M_s \quad \forall\, 0 \leq s \leq t < \infty$.

(2.4) Define: $f: \mathbb{R}^d \to \mathbb{R}_+$ by setting: $f(x) := (x_1^2 + \dots + x_d^2)^{1/2} \equiv \|x\|_2$

so that $D_i f(x) = \dfrac{x_i}{\|x\|_2}$ and $D_i^2 f(x) = -\dfrac{x_i^2}{\|x\|_2^3} + \dfrac{1}{\|x\|_2} = \dfrac{\|x\|_2^2 - x_i^2}{\|x\|_2^3}$

- By Ito chain-rule for $X_t := \|B_t\|_2$ has differential:

$$dX_t = d(f\circ B)_t = \sum_i D_i f(B_t)\, dB_t^i + \frac{1}{2}\sum_{ij} D_{ij}^2 f(B_t)\, \underbrace{dB_t^i dB_t^j}_{=\delta^{ij} dt}$$

$$= \sum_i \frac{B_t^i}{\|B_t\|_2} dB_t^i + \frac{1}{2}\sum_i \frac{\|B_t\|_2^2 - (B_t^i)^2}{\|B_t\|_2^3}\, dt$$

$$= \sum_i \frac{B_t^i}{\|B_t\|_2} dB_t^i + \frac{d-1}{2}\frac{dt}{\|B_t\|_2}$$

- Now we will show that process $Z \equiv (Z_t)_{t\geq 0}$, $Z_t \in \mathbb{R}$ defined by:

$$dZ_t = \sum_i \frac{B_t^i}{\|B_t\|_2} dB_t^i = \sum_i U_t^i\, dB_t^i$$

is also d-dim. B.M.

- We use Lévy characterization: $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_t := \sigma(B_s : s \in [0,t])$ = "smallest σ-algebra that makes $(B_s)_{s\in[0,t]}$ measurable"

- by def. of stochastic integrals

$M_t := \int_0^t F_s\, dB_s$ is a m.g. for any smooth enough $F \equiv (F_s)_{s\geq 0}$ s.t. F_s is $\mathcal{F}_s$-measurable $\forall s$.

- this is easy to check for simple functions: $F_t = \sum R_{t_{j-1}} \chi_{(t_{j-1}, t_j]}(t)$ and then expand to more general integrands by density arguments.

- upshot: $Z_t = \sum_i \int_0^t U_s^i\, dB_s^i \Rightarrow Z$ is m.g.

- Secondly:

$$d(Z^2)_t = 2Z_t dZ_t + \frac{1}{2}\cdot 2\cdot (dZ_t)^2$$

$$= 2Z_t \sum_i U_t^i dB_t^i + \sum_{ij} U_t^i U_t^j dB_t^i dB_t^j$$

- The first term is m.g. so we are left with the second term:

$$\sum_{ij} U_t^i U_t^j \delta_{ij}\, dt = \Big\{\sum_i (U_t^i)^2\Big\} dt = \Big\{\|B_t\|^{-2} \sum_{i=1}^d (B_t^i)^2\Big\} dt = 1\cdot dt.$$

$$\Rightarrow d(Z_t^2 - t) = 2Z_t \sum_i U_t^i dB_t^i \quad \text{a m.g.}$$

$\Rightarrow$ by Lévy's characterization: Z is a B.M.

$$\Rightarrow \boxed{dX_t = dZ_t + \frac{d-1}{2}\frac{dt}{X_t} \qquad (d>1)}$$

- obviously, $\frac{1}{|X_t|}$ causes a repulsion from origin and thus X_t can never become negative even though B.M. Z will eventually become arbitrary large (or small).

(2.5) By definition of stochastic integral:

$$\mathbb{E}(X_t^i) = \mathbb{E}\left[x_0^i + \left(\int_0^t U_s\, dB_s\right)^i\right] = \mathbb{E}\left[0 + \sum_{ij} \int_0^t U_s^{ij}\, dB_s^j\right]$$

$$= \sum_{ij} \underbrace{\mathbb{E}\left[\int_0^t U_s^{ij}\, dB_s\right]}_{=0} = 0$$

$$\mathbb{E}\left[(X_t X_t^T)^{ij}\right] = \mathbb{E}\left[x_t^i x_t^j\right] = \mathbb{E}\left[\left(\sum_k \int_0^t U_s^{ik}\, dB_s^k\right)\left(\sum_l \int_0^t U_v^{jl}\, dB_v^l\right)\right]$$

- Now, we use Ito-isometry: (*)

$$\mathbb{E}\left[\left(\int_0^t V_s\, dB_s^i\right)\left(\int_0^t W_v\, dB_v^j\right)\right] = \left\{\int_0^t \mathbb{E}\left[V_s W_s\right] ds\right\} \delta_{ij}$$

To continue:

$$\mathbb{E}\left[(x_t x_t^T)^{ij}\right] = \sum_{k,l} \mathbb{E}\left[\left(\int_0^t U_s^{ik}\, dB_s^k\right)\left(\int_0^t U_v^{jl}\, dB_v^l\right)\right]$$

$$= \sum_{k,l} \int_0^t \mathbb{E}\left[U_s^{ik} U_s^{jl}\right] ds \cdot \delta_{kl}$$

$$= \int_0^t \mathbb{E}\left[(U_s U_s^T)^{ij}\right] ds = \delta^{ij} \cdot t$$

So $\mathbb{E}(x_t x_t^T) = I \cdot t$. With little extra effort one could now continue and show that $(X_t)_{t \geq 0}$ is d-dim BM.

- The intuitive idea of this result is that since the law of increments $B_t - B_s$ is invariant under rotations the process:

$$\sum_{j=1}^{M} O_{t_{j-1}} (B_{t_j} - B_{t_{j-1}})$$

is still a sum of M i.i.d. Gaussian increments with

$$O_{t_{j-1}} (B_{t_j} - B_{t_{j-1}}) \sim \mathcal{N}(0, (t_j - t_{j-1}) \cdot \mathbb{1})$$

as long as $O_{t_{j-1}}$ and $(B_{t_j} - B_{t_{j-1}})$ are independent.

(*) Ito-isometry:
- for simple functions this is easy to show.
- again for more general case one uses the density of simple functions in the intended process space.

(2.6) WE HAVE:

$$(M, L v)_N = \left(M, \lim_{t \downarrow 0} \frac{1}{t}(T_t v - v)\right)_N$$

$$= \lim_{t \downarrow 0} \frac{1}{t}\left\{ (M, T_t v)_N - (M, v)_N \right\} \quad (1)$$

$\Rightarrow$ SO WE MUST SHOW THAT $T_t v(x) = \int_D P_t(x, dy) v(y)$ IS SYMMETRIC W.R.T. THE STATIONARY MEASURE V.

(a) $$(M, T_t v)_V = \int M(x) T_t v(x) V(dx)$$

$$= \int M(x) \int v(y) P_t(x, dy) V(dx)$$

$$= \iint M(x) v(y) V(dx) P_t(x, dy) \quad (2)$$

TIME-REVERSAL INVARIANCE: $\forall A, B, \forall t \geq 0$

$$\int_A \int_B P_t(x, dy) V(dx) = \mathbb{P}[X_0 \in A, X_t \in B]$$

$$= \mathbb{P}[X_t \in A, X_0 \in B] \quad \text{- choose } k = 2, T = t.$$

$$\equiv \mathbb{P}[X_0 \in B, X_t \in A]$$

$$= \int_B \int_A P_t(x, dy) V(dx)$$

$$\Rightarrow \quad 0 = \left(\int_A\int_B - \int_B\int_A\right) P_t(x, dy) V(dx) = \int_A\int_B \left(P_t(x, dy) V(dx) - P_t(y, dx) V(dy) \right)$$

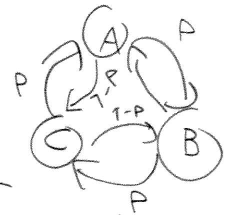
$$\Rightarrow \quad dV(x) P_t(x, dy) = dV(y) P_t(y, dx), \quad \forall x, y, t$$

- USE THIS IN (2)

$$(M, T_t v)_V = \iint M(x) v(y) \, dV(y) P_t(y, dx) = (T_t M, v)_V$$

$\Rightarrow$ SUBSTITUTE INTO (1) WITH $N = V$ $\Rightarrow$ $(M, L v)_V = (L M, v)_V$.

c) IN THERMODYNAMIC EQUILIBRIUM THERE SHOULD BE TIME REVERSAL SYMMETRY.



- ESPECIALLY THIS IMPLIES THAT STATIONARY STATES WHERE THERE ARE NON MEAN ZERO CURRENTS ARE EXCLUDED BY THIS SYMMETRY REQUIREMENT:
- CONSIDER FOLLOWING DISCRETE TIME SYSTEM OF 3 STATES A, B, C WITH TRANSITION RATES $\mathbb{P}[X_j = \beta \mid X_{j-1} = \alpha] = \begin{cases} P, & \beta > \alpha \\ 1-P, & \beta < \alpha \end{cases}$

WHERE $A < B < C < A$

$\Rightarrow$ CLEARLY $\mathbb{P}[X_j = \alpha] = \frac{1}{3}$ IS STATIONARY STATE. BUT FOR $P \neq \frac{1}{2}$ IT IS NOT TIME REVERSAL INVARIANT!

(2.6 b) by Ito's formula:

$$Lu(x) = \lim_{t\to 0} \frac{1}{t}\left\{ \mathbb{E}\left[u(X_t) \mid X_0 = x\right] - u(x)\right\}$$

$$= \lim_{t\to 0} \frac{1}{t}\left\{ \mathbb{E}\left[u(x) + \int_0^t d(u\circ X)_s \,\Big|\, X_0 = x\right] - u(x)\right\}$$

$$= \lim_{t\to 0} \frac{1}{t}\mathbb{E}\left[\int_0^t \left\{u'(X_s)\,dX_s + \frac{1}{2}u''(X_s)(dX_s)^2\right\} \,\Big|\, X_0 = x\right] \quad (1)$$

- using: $dX_s = b(X_s)\,ds + \sigma(X_s)\,dB_s$

$\Rightarrow (dX_s)^2 = \sigma^2(X_s)\,ds$

$$\Rightarrow \mathbb{E}\left[\int_0^t \left\{u'(X_s)\left(b(X_s)ds + \sigma(X_s)dB_s\right) + \frac{1}{2}u''(X_s)\sigma^2(X_s)ds\right\}\,\Big|\, X_0 = x\right]$$

$$= \mathbb{E}\left[\int_0^t \left(b(X_s)u'(X_s) + \frac{1}{2}\sigma^2(X_s)u''(X_s)\right)ds \,\Big|\, X_0 = x\right]$$

$$+ \mathbb{E}\left[\int_0^t u'(X_s)\sigma(X_s)\,dB_s \,\Big|\, X_0 = x\right]$$

$$= \mathbb{E}\left[\int_0^t \tilde{L}u(X_s)\,ds \,\Big|\, X_0 = x\right] + 0, \quad \tilde{L}u(x) := \left(b(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\right)u(x)$$

- substituting this into (1) gives

$$Lu(x) = \lim_{t\to 0}\frac{1}{t}\int_0^t \mathbb{E}\left[\tilde{L}u(X_s) \mid X_0 = x\right]ds = \tilde{L}u(x)$$

$$\equiv \left(b(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\right)u(x)$$

- Let's now find conditions on b, σ so that X is time-reversible w.r.t. $\nu = \text{Leb}$: $(u,v) := (u,v)_{\text{Leb}}$

$$(u, Lv) = \int_{\mathbb{R}} u\cdot\left[\frac{1}{2}\sigma^2 v'' + b v'\right]dx$$

$$= \int_{\mathbb{R}}\left\{-v'\frac{\partial}{\partial x}\left(\frac{1}{2}\sigma^2 u\right) - v\frac{\partial}{\partial x}(b\cdot u)\right\}dx + \Big/_{-\infty}^{\infty}\left\{v'\cdot\left(\frac{1}{2}\sigma^2 u\right) + v\cdot b\cdot u\right\}$$

$$= \int_{\mathbb{R}}\Big[\underbrace{\frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 u) - \frac{\partial}{\partial x}(b\cdot u)}_{=:L^* u}\Big]v\,dx + \Big/_{-\infty}^{\infty}(\cdot\cdot) + \Big/_{-\infty}^{\infty}(\cdot\cdot)$$

- We assume that the space of which u, v are taken is such that boundary terms disappear: for example we could require them to have a compact support.

- Conditions on b, σ: set $a(x) := \sigma^2(x)$:

$$L^* u = \frac{1}{2}a\cdot u'' + a'u' + \frac{1}{2}a''u - bu' - b'u$$

$$= \left[\left(\frac{1}{2}a\frac{\partial^2}{\partial x^2}\right) + (a' - b)\frac{\partial}{\partial x} + \left(\frac{1}{2}a'' - b'\right)\right]u$$

$\Rightarrow$ match terms with L: $\frac{\partial}{\partial x}$: $a' - b = b \Leftrightarrow a' = 2b \Leftrightarrow \underline{\frac{1}{2}\frac{\partial}{\partial x}(\sigma^2) = b}$

$$\Rightarrow Lu = \frac{1}{2}a\cdot\frac{\partial^2}{\partial x^2}u + \frac{1}{2}\left(\frac{\partial}{\partial x}a\right)\frac{\partial}{\partial x}u = \frac{1}{2}\frac{\partial}{\partial x}\left(a(x)\frac{\partial}{\partial x}u\right)$$