Quantum Probability: Exercise set 2 (10.2.2010)

We denote $\mathbb{R}_+ := [0, \infty)$. A process $(X_t : t \in \mathbb{R}_+)$ is denoted by (X_t) or simply X. As usual we assume everything is defined on a probability space $(\Omega, \Sigma, \mathsf{P})$ unless otherwise stated. A sequence $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{R}_+)$ of sub-σ-algebras of Σ that is increasing w.r.t. time, i.e., $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$, is called a filtration. If (X_t) satisfies $\sigma(X_t) \subset \mathcal{F}_t$ for every $t \geq 0$ then we say that (X_t) is adapted to $\mathbb{F}$. A d-dimensional Brownian Motion, is a process (B_t) on $\mathbb{R}^d$ where each component $B^j = (B^j_t : t \in \mathbb{R}_+)$ is an independent (1-dimensional) Brownian Motion (BM).

1. By using Kolmogorov's continuity theorem and the finite dimensional distributions of BM, show that there exists a continuous version of BM.

2. Suppose $\lambda > 0$ is a constant and (B_t) is a BM. (a) Show the process (X_t) defined by $X_t := \lambda^{-1} B_{\lambda^2 t}$ is also BM; (b) calculate the correlation $\mathsf{E}(X_s B_t)$.

3. Let B be a BM. (a) Prove directly from the definition of Ito-integral that

$$\int_0^t B_s^2 \mathrm{d}B_s = \frac{1}{3} B_t^3 - \int_0^t B_s \mathrm{d}s \,.$$

   (b) Verify this by applying Ito-chain rule to the right side.

4. Let $|\bullet|$ be a Euclidean norm in $\mathbb{R}^d$ and B a d-dimensional BM. Find the stochastic differential equation (SDE) satisfied by (X_t) when $X_t := |B_t|$.

5. Let B be a d-dimensional $\mathbb{F}$-adapted BM. Let (U_t) be a $\mathbb{R}^{d \times d}$-matrix valued $\mathbb{F}$-adapted continuous stochastic process that satisfies $\mathsf{P}\big[U_t U_t^{\mathrm{T}} = I\big] = 1$ for every $t \in \mathbb{R}_+$, where I is the identity matrix. Define $X_t = (X_t^1, \ldots, X_t^d)$ by setting $\mathrm{d}X_t^i = \sum_j U_t^{ij} \mathrm{d}B_t^j$, $X_0 = 0$. Prove that $\mathsf{E}(X_t) = 0$ and $\mathsf{E}(X_t X_t^{\mathrm{T}}) = tI$ by using the fact that $\mathsf{E}(\int_0^t A_s \mathrm{d}B_s) = 0$ for all $\mathbb{F}$-adapted processes encountered here (technically speaking A should be, for example, a previsible process). By using Lévi-characterization of BM one can use this result to show that X is also a d-dimensional BM.

6. Let $X = (X_t)$ be a time-homogenous Markov process on some domain $D \subset \mathbb{R}^d$. Suppose ν is the stationary measure of X, i.e., $S_t \nu = \nu$, where S_t is the dual operator of T_t, i.e., $Tu(x) = \int_D p_t(x, \mathrm{d}y)(y)$ and $S\mu = \int_D \mu(\mathrm{d}x) p_t(x, \bullet)$. Now, suppose that X is time-reversible, i.e., if X starts from its stationary distribution $\mathsf{P}(X_0 \in \mathrm{d}x) = \nu(\mathrm{d}x)$ then $\mathsf{P}(X_{t_1} \in B_1, \ldots, X_{t_k} \in B_k) = \mathsf{P}(X_{T - t_k} \in B_k, \ldots, X_{T - t_1} \in B_1)$ for any times $0 \leq t_{i-1} \leq t_i \leq T < \infty$ and Borel sets B_j, $k \in \mathbb{N}$.

   (a) Define an inner product $(u, v)_\mu := \int_D u(x) v(x) \mu(\mathrm{d}x)$ w.r.t. a positive measure μ. Show that time reversibility of X implies that the generator

$$Lu := \lim_{\tau \searrow 0} \frac{1}{\tau} (T_\tau u - u) \,,$$

   must be self adjoint w.r.t. invariant measure ν, i.e., $(u, Lv)_\nu = (Lu, v)_\nu$.

   (b) Now, let $D = \mathbb{R}^1$ and suppose that X satisfies SDE: $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \gamma(X_t)\mathrm{d}W_t$. What conditions must $b, \gamma : \mathbb{R} \to \mathbb{R}$ satisfy for X to be time reversible w.r.t. Lebesgue measure (note that this is not a probability measure).

   (c) What has this exercise to do with thermodynamics? (this is for physicists).