Quantum Probability: exercise session 1 (for session on 3.2.2010)

Don't mind too much if you are not sure about all the details!

Typically the abstract probability space is denoted by $(\Omega, \mathcal{F}, \mathsf{P})$. Unless otherwise stated, we assume that a r.v. $X \to S$, where S is a metric space, is measurable w.r.t. Borel σ algebra $\mathcal{B}(S)$ corresponding the metric topology of S, i.e., $X^{-1}(B) \in \mathcal{F}$ for any $B \in \mathcal{B}(S)$. Note that there are many ways to write the same things:

$$\mathsf{P}_X(A) \equiv \mathsf{P} \circ X^{-1}(A) \equiv \mathsf{P}(X \in A) \equiv \mathsf{P}(\{\omega \in \Omega : X(\omega) \in A\}) .$$

Recall also that $\sigma(X) := \{X^{-1}(B) : B \in \mathcal{B}(S)\}$. Since E is a linear operator we often drop the parenthesis, e.g., $\mathsf{E}u(X) \equiv \mathsf{E}[u(X)] \equiv \int_\Omega u(X(\omega))\mathsf{P}(\mathrm{d}\omega)$. We denote by χ_A the indicator function the event A, e.g., $\chi_A : \Omega \to \{0,1\}$ with $\chi_A(\omega) = 1$ if and only if $\omega \in A$.

1. Let $f : S \to T$ be an arbitrary function between two sets S and T. Recall, that $f^{-1}(B) := \{s \in S : f(s) \in B\}$, for $B \subset T$. Proof that f^{-1} preserves the set operations in the sense that for any subsets B and B_k of T the following hold: (a) $f^{-1}(B^{\mathrm{c}}) = (f^{-1}(B))^{\mathrm{c}}$; (b) $f^{-1}(\cup_k B_k) = \cup_k f^{-1}(B_k)$; (c) $f^{-1}(\cap_k B_k) = \cap_k f^{-1}(B_k)$.

2. Let $\{\mathcal{G}_\alpha : \alpha \in A\}$ be a family of σ-algebras. Show that $\mathcal{G} := \cap_{\alpha \in A} \mathcal{G}_\alpha$ is σ-algebra. This result implies that $\sigma(X)$, the smallest σ-algebra, in which r.v. X is measurable always exists.

3. Let $\mathcal{R}$ be a finite partition of Ω. Let $\mathcal{G}$ consists of the empty set $\emptyset$, and all possible unions of $R \in \mathcal{R}$. Show that (a) $\mathcal{G}$ is a σ-algebra; (b) Any finite σ-algebra is of this type; (c) Let $X : \Omega \to \mathbb{R}$ be $\mathcal{G}$-measurable r.v. Proof that X can take only finitely many values $\{x_j : 1 \leq j \leq N\} \subset \mathbb{R}$, $N \in \mathbb{N}$, and express X is terms of χ_G, $G \in \mathcal{G}$.

4. Let X, Y be random variables. (a) Suppose they both can take only finitely many values. Express $\sigma(X, Y)$ in terms of the elements of $\sigma(X)$ and $\sigma(Y)$. (b) Now, suppose X is general, but Y can take finitely many values. Write $\mathsf{E}(X|Y) := \mathsf{E}(X|\sigma(Y))$ as a deterministic function $f(Y)$ of Y.

5. (a) Suppose $\phi : \mathbb{R} \to \mathbb{R}_+$ is increasing function. Proof that for $\mathbb{R}$-valued r.v. X one has:
$$\mathsf{P}(X \geq x) \leq \frac{\mathsf{E}\phi(X)}{\phi(x)} \qquad \text{for every} \quad x \in \mathbb{R} .$$
(b) Assume that $\mathsf{E}\mathrm{e}^{\alpha|X|} < \infty$, for some $\alpha > 0$. Show that $\mathsf{P}(|X| \geq r) \leq M\mathrm{e}^{\lambda r}$, for some $M < \infty$.

6. Construct random variables X, Y, Z such that the pairs $\{X, Y\}, \{X, Z\}$ and $\{Y, Z\}$ are independent, but $\{X, Y, Z\}$ are not independent.

7. Suppose $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, and two random variables satisfy $\mathsf{E}|XY| < \infty$. Show that (a) if X is $\mathcal{G}$-measurable then $\mathsf{E}(XY|\mathcal{G}) = X\mathsf{E}(Y|\mathcal{G})$; (b) if X is independent of $\mathcal{G}$ then $\mathsf{E}(X|\mathcal{G}) = \mathsf{E}X$; (c) if $\tilde{\mathcal{G}} \subset \mathcal{G}$ is another coarser sub-σ-algebra then $\mathsf{E}(X|\tilde{\mathcal{G}}) = \mathsf{E}(\mathsf{E}(X|\mathcal{G})|\tilde{\mathcal{G}})$. *Hint:* Use simple functions.