Quantum Probability Home Exam 2010 (Version 1.1)
- Due to Monday 24.5.2010, 12.00 o'clock.

Write one problem per sheet. Write enough intermediate steps and try to be as clear as possible. If you suspect you have found an error, please send email to: `oskari.ajanki@iki.fi`

Remark: This exam is somewhat demanding but not as demanding as it may appear at the first glance. You may find hints and other support information provided with the problems.

1. Let $\{B_t : t \geq 0\}$ be a (one-dimensional) Brownian motion and set

$$K_t := \alpha t + B_t\,, \tag{0.1}$$

where $\alpha \in \mathbb{R}$ is a constant. Set $(X_t, Y_t) = (\cos K_t, \sin K_t)$.

(a) Find stochastic differential equation for (X_t, Y_t), i.e., find functions $b : \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ and a 2-dimensional Brownian motion $\{W_t : t \geq 0\}$, with $W_t = (W_t^1, W_t^2)$, such that $(\mathrm{d}X_t, \mathrm{d}Y_t) = b(X_t, Y_t)\mathrm{d}t + \sigma(X_t, Y_t)\mathrm{d}W_t$

(b) Find the Markov semigroup generator $\mathcal{L}$ so that

$$(\mathrm{e}^{t\mathcal{L}}u)(x, y) \;=\; \mathsf{E}\big[u(X_t, Y_t)\big|(X_0, Y_0) = (x, y)\big]\,,$$

for an appropriate class of functions $u : \mathbb{R}^2 \to \mathbb{R}$.

(c) Find the stationary state measure for (X_t, Y_t). When does the detailed balance hold?

(d) How do the answers to (b) and (c) change if instead of (0.1) the process K_t is the solution of the stochastic differential equation

$$\begin{aligned} \mathrm{d}K_t &= \alpha\mathrm{d}t + X_t\mathrm{d}B_t - Y_t\mathrm{d}B'_t \\ K_0 &= 0\,, \end{aligned}$$

where $\{B_t\}$ and $\{B'_t\}$ are independent copies standard Brownian motions.

2. In this problem we consider a mechanical chain of n oscillators connected to stochastic heat baths at the both ends. Let $q_x \in \mathbb{R}$ be the deviation of the oscillator $x = 1, \ldots, n$ from its equilibrium position ($q_x = 0$) and let p_x be the corresponding momentum. The Hamiltonian for the system is

$$H(p, q) = \sum_{x=0}^{n} \left\{ \frac{p_x^2}{2} + v(q_x) + u(q_{x+1} - q_x) \right\} \qquad \text{with} \qquad q_0 = q_{n+1} := 0\,,$$

where $u, v : \mathbb{R} \to [0, \infty)$ are smooth potential functions which grow monotonically as one moves away from the origin and $u(0) = v(0) = 0$. Let $T_1 \geq T_n > 0$ be temperatures of the left ($x = 1$) and right ($x = n$) heat baths, respectively. We model the coupling of the oscillators to the heat baths by adding noise to the Hamiltonian equations

$$\begin{aligned} \mathrm{d}q_x &= \frac{\partial H}{\partial p_x}(p, q)\mathrm{d}t \\ \mathrm{d}p_x &= -\frac{\partial H}{\partial q_x}(p, q)\mathrm{d}t \;+\; (\delta_{x1} + \delta_{xn})(-\lambda p_x \mathrm{d}t + \sqrt{2\lambda T_x}\mathrm{d}w_x)\,, \end{aligned} \tag{0.2}$$

where $w_x = \{w_x(t) : t \geq 0\}$, $x = 1, n$, are independent Brownian motions, and $\lambda > 0$ is a coupling constant.

(a) Let $\mathcal{E}(t) := H(p(t), q(t))$ be the energy of the system at time t. By using Ito-calculus and the fact that Hamiltonian dynamics conserve energy one can write

$$\mathrm{d}\mathcal{E} \;=\; j_1(p_1)\mathrm{d}t \;-\; j_n(p_n)\mathrm{d}t \;+\; g_1(p,q)\mathrm{d}w_1 \;+\; g_n(p,q)\mathrm{d}w_n\,, \tag{0.3}$$

where $j_x : \mathbb{R} \to \mathbb{R}$ and $g_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Find explicit formulas for the functions j_1, j_n, g_1, g_n.

(b) Suppose that $T_1 = T_n = T > 0$. Show that the Gibbs-measure

$$\mu_T(\mathrm{d}p\,\mathrm{d}q) = Z_T^{-1}\mathrm{e}^{-\frac{1}{T}H(p,q)}\mathrm{d}p\,\mathrm{d}q \qquad \text{with} \qquad Z_T = \iint \mathrm{e}^{-\frac{1}{T}H(p,q)}\mathrm{d}p\,\mathrm{d}q\,,$$

is the stationary state probability measure of $\{(p(t), q(t)) : t \geq 0\}$.

(c) Show that

$$\int j_x(p_x)\mathrm{d}\mu_T(\mathrm{d}p\,\mathrm{d}q) \;=\; 0 \qquad \text{for} \quad x = 1, n\,.$$

Suppose μ_{T_1,T_n} is the general ergodic stationary state probability measure of (0.2), e.g., $\mu_{T,T} = \mu_T$. Use this to proof that

$$\int j_1(p_1)\mathrm{d}\mu_{T_1,T_n}(\mathrm{d}p\,\mathrm{d}q) \;=\; \int j_n(p_n)\mathrm{d}\mu_{T_1,T_n}(\mathrm{d}p\,\mathrm{d}q) \;=:\; \bar{J}(T_1, T_n)\,.$$

What is the physical interpretation of the quantity $\bar{J}(T_1, T_n)$?

3. Consider a ideal Bose gas in the box $V := [-L, L]^d$. In the coordinate basis the system is thus described by a wave function $\psi \in \Gamma_s(L^2(V, m_d))$ where m_d is the d-dimensional Lebesgue measure. Let $\{\phi_j : j \in \mathbb{N}_0\}$ be the eigenfunctions of the single particle Hamiltonian h so that in the coordinate representation $h\phi_j = -\Delta\phi_j = \epsilon_j\phi_j$ with $\phi_j(x) = |V|^{-1/2}\mathrm{e}^{\mathrm{i}k_j\cdot x}$ and $\epsilon_j = |k_j|^2$. Denote the Weyl operator $W(f) := \mathrm{e}^{\mathrm{i}\Phi(f)}$ where $\Phi(f) := 2^{-1/2}(a^\dagger(f) + a(f))$ (Note: different normalization of Φ was used in the lecture notes). The generating function corresponding to the density operator ρ is defined by $E_\rho(f) := \mathrm{Tr}\,\{W(f)\rho\}$ where f is taken from some appropriate test function space.

   (a) Compute the generating function $E_\Omega(f) = \langle\Omega, W(f)\Omega\rangle$ of the vacuum state.

   (b) Let $n_0 \in \mathbb{N}_0$ be the number of particles in the ground state and denote by $\theta_0 = n_0/V$ the density of particles in the ground state. Show that the corresponding generating function is

$$E_{V,\theta_0}(f) \;=\; E_\Omega(f)\cdot\frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l}\frac{n!}{(n-l)!}\left(-\frac{|\langle\phi_0, f\rangle|^2}{2}\right)^{n-l}.$$

   By assuming that the support of f is bounded (compact) and using the known limit

$$j_0(2\sqrt{z}) \;=\; \lim_{n\to\infty}\frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l}\frac{n!}{(n-l)!}\left(\frac{-z}{n}\right)^{n-l},$$

   where j_0 is a Bessel function, show that in the infinite volume limit $|V| \to \infty$ one obtains

$$E_{\theta_0}(f) \;:=\; \lim_{|V|\to\infty} E_{V,\theta_0}(f) \;=\; E_\Omega(f)\cdot j_0\big((2\pi)^{d/2}(2\theta_0)^{1/2}|\hat{f}(0)|\big)\,,$$

   where $\hat{f}(k) := (2\pi)^{-d/2}\int \mathrm{e}^{-\mathrm{i}k\cdot x}f(x)\mathrm{d}x$.

(c) Let $\theta_j = n_j/|V|$, $|V| \equiv m_d(V) = (2L)^d$, denote the density of particles which are in the j:th energy state ϕ_j and consider an excited state

$$\Psi_{V,\boldsymbol{\theta}} = \frac{1}{\sqrt{n_1! \cdots n_\ell!}} a^\dagger(\phi_1)^{n_1} \cdots a^\dagger(\phi_\ell)^{n_\ell} \Omega \,,$$

where $\boldsymbol{\theta} \equiv (\theta_0, \theta_1, \ldots, \theta_\ell)$ (Note: ϕ_j depends on the box size V even thought this is not indicated by the notations.). Let $E_{V,\boldsymbol{\theta}}(f) = \langle \Psi_{V,\boldsymbol{\theta}}, W(f) \Psi_{V,\boldsymbol{\theta}} \rangle$ denote the finite volume generating function. Using using the same ideas as in the part (b) show that the infinite volume generating function $E_{\boldsymbol{\theta}}(f) := \lim_{|V| \to \infty} E_{V,\boldsymbol{\theta}}(f)$ is

$$E_{\boldsymbol{\theta}}(f) \;=\; E_\Omega(f) \cdot \prod_{j=0}^{\ell} j_0\big((2\pi)^{d/2} (2\theta_j)^{1/2} |\hat{f}(k_j)|\big) \,. \tag{0.4}$$

(d) In the infinite volume limit, the values k_j in (0.4) are not restricted by boundary conditions and can therefore take values in the continuum, i.e., in $\mathbb{R}^d$. Thus, as the next step it is natural to generalize formula (0.4) for continuous distribution $k \mapsto \theta(k)$ of momentum values. Show that

$$E_\theta(f) \;=\; \exp\left[-\frac{1}{4} \langle \hat{f}, (1 + 2(2\pi)^d \theta) \hat{f} \rangle \right] \,.$$

Hint: Consider first a fixed momentum range $[-K, K]^d$, $K > 0$, in (0.4). Let $\boldsymbol{\theta}^{(K,m)}$, $m \in \mathbb{N}$, be a density vector such that the component $\theta_j^{(K,m)}$, $j \equiv (j_1, \ldots, j_d) \in \{0, 1, \ldots, m\}^d$ describes the density of particles with a momentum $k = (k_1, \ldots, k_d)$, $k_i = -K + \frac{2K}{m} j_i$. Use the estimate

$$\ln j_0(\epsilon) \;=\; -\epsilon^2/2 \,+\, \mathcal{O}(\epsilon^{2+\delta}) \qquad \text{for} \qquad \delta > 0 \,,$$

and the above (K, m)-discretion scheme to express (0.4) as an exponent of an inner product. Notice that you have to take two limits.

(e) Finally, find out the generating function $E_{\theta_0,\theta}(f)$ for the infinitely extended ideal Bose-gas where $\theta_0 \geq 0$ fraction of the particles are at the ground state, while the rest of the particles are distributed according to the density $\theta : \mathbb{R}^d \to [0, \infty)$. For which choices of θ_0, θ does this correspond as stationary state under the Hamiltonian evolution?

Motivation: The reason one considers the generating functions instead of, for example the usual Fock-representation, is the fact that the canonical Fock space does not support states ψ for which $\bar{N} := \langle \psi, N \psi \rangle = \infty$. When $\int \theta(x) \mathrm{d}x > 0$ and $|V| = \infty$ the states considered in this exercise obviously yield $\bar{N} = \infty$ and therefore Fock representation can not be used.

4. (a) Suppose $\mathcal{L}_k$, $k = 1, 2$, are generators of dynamical semigroups, and let $\mathcal{L} := \mathcal{L}_0 + \mathcal{L}_1$. Show that

$$\begin{aligned} \mathrm{e}^{t\mathcal{L}} \;&=\; \mathrm{e}^{t\mathcal{L}_0} \,+\, \int_0^t \mathrm{e}^{(t-s)\mathcal{L}} \mathcal{L}_1 \mathrm{e}^{s\mathcal{L}_0} \mathrm{d}s \\ &=\; \mathrm{e}^{t\mathcal{L}_0} \,+\, \int_0^t \mathrm{e}^{(t-s)\mathcal{L}_0} \mathcal{L}_1 \mathrm{e}^{s\mathcal{L}} \mathrm{d}s \,. \end{aligned}$$

Note: Generally $(\mathcal{L}A)B \neq \mathcal{L}(AB) \equiv \mathcal{L}AB$.

(b) It is well known that the Lindblad representation

$$\mathcal{L}(\rho) := -\mathrm{i}[H, \rho] \,+\, \frac{1}{2} \sum_\alpha \big(2L_\alpha \rho L_\alpha^\dagger - L_\alpha^\dagger L_\alpha \rho - \rho L_\alpha^\dagger L_\alpha \big)$$

of the generator $\mathcal{L}$ of a completely positive semigroup is not unique. Show that the self-adjoint operator H and Lindblad operators L_α can be chosen such that $\operatorname{Tr} L_\alpha = 0$ for every α.

5. Consider the final section, *Quantum Brownian motion*, of the lecture notes. Show that the particle travels ballistically, i.e., $\operatorname{Tr}(X^2\rho_t) \sim t$ provided we set $\lambda = 0$.

6. Consider a system consisting of N qubits $\mathcal{H}_S := (\mathbb{C}^2)^{\otimes N}$, and an environment consisting of bosonic field modes $\mathcal{H}_R = \Gamma_s(\ell^2(\mathbb{N}))$. Let the total Hamiltonian be

$$\begin{aligned} H &\equiv H_S + H_E + H_I \\ &= \sum_{j=0}^{N-1} h_j \otimes 1_E + 1_S \otimes \sum_k \omega_k a_k^\dagger a_k + \sum_{j=0}^{N-1} Z_j \otimes \sum_k (g_{jk} a_k^\dagger + g_{jk}^* a_k), \end{aligned}$$

where Z is the Pauli spin matrix in the z-direction, $Z_j := 1^{\otimes j-1} \otimes Z \otimes 1^{\otimes(n-j-1)}$ is Z applied to j:th qubit, $h_j := \varepsilon_j Z_j$ with $\varepsilon_j > 0$, $\omega_k \geq 0$ and $g_k \in \mathbb{C}$. As usual we set $H_0 = H_S + H_E$ and simplify notations by writing $H_S \equiv H_S \otimes 1_E$, $H_E \equiv 1_S \otimes H_E$, $\sigma_{j,z} \equiv \sigma_{j,z} \otimes 1_E$, etc. Let us write $\tilde{A}(t) = \mathrm{e}^{\mathrm{i}H_0 t} A \mathrm{e}^{-\mathrm{i}H_0 t}$ for a general observables A and $\tilde{\rho}(t) := \mathrm{e}^{-\mathrm{i}H_0 t} A \mathrm{e}^{\mathrm{i}H_0 t}$ for density operators on $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$.

Let us start with a *special case* $N = 1$ (you may drop the j-index from Hamiltonian) and generalize the results for $N \geq 2$ later.

(a) Show that $[\tilde{H}_I(t), \tilde{H}_I(s)] = h(t-s) 1_{SE}$ and find out function $h : \mathbb{R} \to \mathbb{C}$.

(b) For a time dependent operator A show that $\left[A(t), \frac{\mathrm{d}}{\mathrm{d}t} A(t)\right] = c(t) I$ implies $\frac{\mathrm{d}}{\mathrm{d}t}(A^n) = n A^{n-1} \frac{\mathrm{d}}{\mathrm{d}t} A - \frac{1}{2} n(n-1) c A^{n-2}$.

(c) Denote $\tilde{U}(t) := \mathrm{e}^{-\mathrm{i}\tilde{H}_I(t)}$ and use parts (a) and (b) to write

$$\tilde{U}(t) = \mathrm{e}^{\mathrm{i}\phi(t)} \exp\left[-\mathrm{i} \int_0^t \tilde{H}_I(s) \mathrm{d}s\right]. \tag{0.5}$$

Express the phase function $\phi : \mathbb{R} \to \mathbb{R}$ in terms of the function h.

(d) Assume the system and the environment are initially uncorrelated, $\rho_{SE}(0) = \rho_S \otimes \rho_E$, so that $\tilde{\rho}_S(t) := \operatorname{Tr}_E\{\tilde{U}(t) \rho_S \otimes \rho_E \tilde{U}(t)^\dagger\}$. Write (Note: $N = 1$ here)

$$\langle m|\tilde{\rho}_S(t)|n\rangle = \langle m|\rho_S|n\rangle \cdot \chi_{mn}(t), \qquad \text{with} \quad m, n \in \{0,1\}^N, \tag{0.6}$$

and express the non-diagonal ($m \neq n$) suppression factors in the form

$$\chi_{mn}(t) = \operatorname{Tr}\{\rho_E \prod_k D_k(\alpha_k(t))\} \qquad \text{where} \qquad D_k(z) := \mathrm{e}^{z a_k^\dagger + z^* a_k}, \tag{0.7}$$

$z \in \mathbb{C}$, and $\alpha_k(t)$ are explicitly solved analytic function of the parameters $\{\omega_l\}$ and $\{g_k\}$ (recall $g_k \equiv g_{0k}$) for all times $t \in \mathbb{R}$. What about the diagonal suppression factors - can their behavior be deduced without the explicit solution?

(e) Compute $\chi_{mn}(t)$ when $\rho_E = |\Omega\rangle\langle\Omega|$ where Ω is the vacuum state state of the environment: $a_k \Omega = 0$, $k \in \mathbb{N}$. How does $\chi_{mn}(t)$ behave for small times?

(f) Suppose environment starts from the Gibbs state $\rho_E = Z_\beta^{-1} \mathrm{e}^{-\beta H_E}$, $\beta > 0$. Show that

$$\chi_{mn}(t) = \exp\left[-\sum_k \frac{|\alpha_k(t)|^2}{2} \coth\left(\frac{\beta\omega_k}{2}\right)\right].$$

(g) Let us now move to the general case $N \geq 2$: Even though, $[\tilde{H}_I(t), \tilde{H}_I(s)]$ is not a multiple of identity for arbitrary couplings $\{g_{jk}\}$ it is still possible to generalize (0.5) for $N \geq 2$. Show that

$$\tilde{U}(t) \;=\; (\tilde{V}(t) \otimes 1_E)\mathrm{e}^{-\mathrm{i}\bar{H}(t)}\,,$$

where $\bar{H}(t) := \int_0^t \tilde{H}_I(s)\mathrm{d}s$ and $[\tilde{V}(t), Z_j] = 0$ for every $j = 0, \dots, N-1$.

(h) Suppose $N = 2$ and assume the only non-zero couplings are g_{j1} for $j = 0, 1$. Let $\phi := c_{00}|0,0\rangle + c_{11}|1,1\rangle$ and $\psi = c_{10}|1,0\rangle + c_{01}|0,1\rangle$, where $\mathbb{C}^2 = \mathrm{span}(\{|0\rangle, |1\rangle\})$. Calculate $\tilde{U}(t)\phi \otimes \Omega$ and $\tilde{U}(t)\psi \otimes \Omega$. Express them w.r.t. coherent states of the field.

(i) Write $m \equiv (m_0, m_2, \dots, m_{N-1})$, $n \equiv (n_0, n_2, \dots, n_{N-1}) \in \{0,1\}^N$. Find a general suppression factor $\chi_{mn}(t)$ such that (0.6) holds. Express the result in the form analogous to (0.7) so that functions $\alpha_{jk}(t)$ analogous to $\alpha_k(t)$ appear, etc.

(j) Suppose that all the qubits experience the same electromagnetic field, e.g., the case where couplings are independent of j: $g_{jk} = g_k$. This is the idealization of the case where qubits are physically so close to each other that they all see the same EM-field. Decompose the space $\mathcal{H}_S$ into a direct sum

$$\mathcal{H}_S \;=\; \bigoplus_q \mathcal{H}_S^{(q)}\,,$$

of subspaces $\mathcal{H}_S^{(q)}$ which have the following property: $\langle \psi_1, \tilde{\rho}_S(t)\psi_2\rangle = \langle \psi_1, \rho_S \psi_2\rangle$ for every $t \geq 0$ if and only if there exists q such that $\psi_1, \psi_2 \in \mathcal{H}^{(q)}$. What is the maximum dimension of these decoherence free subspaces, e.g., find out $\max_q \dim(\mathcal{H}_S^{(q)})$ for given N? On the other hand, what is the maximum decay rate $\max_{m,n} \chi_{mn}(t)$ (consider small times) and how does it compare to the decay rate in the case of $N = 1$?
Hint: To get an idea of what is going on here consider the exercise (h) with identical couplings.

(k) Finally, let us consider the other extreme scenario where each qubit is coupled to its own field, e.g., they are so far apart physically that they see complete different environments. The simplest way to model this situation is to set $g_{jk} = \delta_{kj}$. Suppose ρ_E is the Gibbs state (See: part (f)). Express $\chi_{mn}(t)$ in the form

$$\chi_{mn}(t) = \mathrm{e}^{-\|m-n\|_H F_\beta(t)}\,,$$

where $\|m\|_H := \sum_{j=0}^{N-1} |m_j|$ is the Hamming distance. How much faster is the decay of $\chi_{mn}(t)$ compared to the single qubit case in the worst case?

7. Let N_t be a Poisson process with parameter $\lambda > 0$ and let T_n be the n jump time, i.e., $T_n := \inf\{t \geq 0 : N_t = n\}$. Show that the random vector $\big(T_1/T_{n+1}, T_2/T_{n+1}, \dots, T_n/T_{n+1}\big)$ has the same probability distribution as $\big(V_1^{(n)}, V_2^{(n)}, \dots, V_n^{(n)}\big)$ where $V_k^{(n)}$ is the k smallest number in the set $\{U_1, \dots, U_n\}$ of independent and uniformly on $(0,1)$ distributed random variables U_j.

8. Consider a generator

$$\mathcal{L}(\rho) := -\mathrm{i}[H, \rho] \,+\, \sum_\alpha \gamma_\alpha \big(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2} L_\alpha^\dagger L_\alpha \rho - \frac{1}{2} \rho L_\alpha^\dagger L_\alpha\big)\,,$$

of a completely positive dynamical semigroup. Define super operators $\{\mathcal{L}_\alpha : \alpha = 0, 1, 2, \dots, N\}$ by $\mathcal{L}_\alpha \rho := \gamma_\alpha L_\alpha \rho L_\alpha^\dagger$ for $\alpha \geq 1$ and $\mathcal{L}_0 := \mathrm{i}[H_{\mathbb{C}}, \bullet]$ where $H_{\mathbb{C}} := H - \frac{\mathrm{i}}{2}\sum_\alpha \gamma_\alpha L_\alpha^\dagger L_\alpha$. By these definitions it follows that $\mathcal{L} = \mathcal{L}_0 + \mathcal{S}$ with $\mathcal{S} = \sum_{\alpha \geq 1} \mathcal{L}_\alpha$.

(a) Derive the representation

$$\mathrm{e}^{t(\mathcal{L}_0+\mathcal{L}_1)} = \mathrm{e}^{t\mathcal{L}_0} + \sum_{n=1}^{\infty}\int_{t\geq t_n\geq\cdots\geq t_1\geq 0}\sum_{\alpha_1,\dots,\alpha_n=1}^{N} \mathrm{e}^{(t-t_n)\mathcal{L}_0}\mathcal{L}_{\alpha_n}\mathrm{e}^{(t_n-t_{n-1})\mathcal{L}_0}\mathcal{L}_{\alpha_{n-1}}\mathrm{e}^{(t_{n-1}-t_{n-2})\mathcal{L}_0}\cdots\mathrm{e}^{(t_2-t_1)\mathcal{L}_0}\mathcal{L}_{\alpha_1}\mathrm{e}^{t_1\mathcal{L}_0}\,\mathrm{d}t_1\cdots\mathrm{d}t_n\,.$$

Let $\omega \equiv \{(t_j,\alpha_j) : j\in\mathbb{N}\} \subset [0,\infty)\times\{1,2,\dots,N\}$. Denote the set of such ω for which $t_j < t_{j+1}$, $j\in\mathbb{N}$, and the number of times t_j on any finite interval $[s,t]$ is finite by Ω. Let us define a super operator for each $\omega\in\Omega$ and $t\geq 0$ by setting:

$$\mathcal{W}_t^{\omega}(\rho) := \frac{\widetilde{\mathcal{W}}_t^{\omega}\rho}{\mathrm{Tr}\big[\widetilde{\mathcal{W}}_t^{\omega}\rho\big]}$$

where

$$\widetilde{\mathcal{W}}_t^{\omega}(\rho) := \mathrm{e}^{(t-t_n)\mathcal{L}_0}\mathcal{L}_{\alpha_n}\mathrm{e}^{(t_n-t_{n-1})\mathcal{L}_0}\mathcal{L}_{\alpha_{n-1}}\mathrm{e}^{(t_{n-1}-t_{n-2})\mathcal{L}_0}\cdots\mathrm{e}^{(t_2-t_1)\mathcal{L}_0}\mathcal{L}_{\alpha_1}\mathrm{e}^{t_1\mathcal{L}_0}\rho\,,$$

and n is the largest integer j in the definition of ω such that $t_j\leq t$. The idea behind all these definitions is that one may consider $\omega\mapsto\mathcal{W}_t^{\omega}$ as a super operator valued random variable on the probability space $(\Omega,\mathcal{F},\mathsf{P})$ where the probability measure P is defined by the expression we derived in (a). We do not bother to try to specify the σ-algebra $\mathcal{F}$ here.

(b) Show that $\mathcal{W}_t^{\omega}(\rho)$ is (i) completely positive and (ii) non-linear.

(c) What is the interpretation of $\mathrm{Tr}\,[\mathrm{e}^{t\mathcal{L}_0}\rho]$? Show that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\,[\mathrm{e}^{t\mathcal{L}_0}\rho] < 0$.

(d) Finally, write $\mathcal{W}_t^{\omega}$ into form

$$\mathcal{W}_t^{\omega}(\rho) = \frac{M_t^{\omega}\rho M_t^{\omega}}{\mathrm{Tr}\left[(M_t^{\omega})^{\dagger}M_t^{\omega}\rho\right]}\,.$$

This shows us that in the light of general measurement theory one may view $\mathcal{W}_t^{\omega}(\rho)$ as a hypothetical random measurement of the the system by the environment.