Computational Statistics
2. exercise session, 5.2.2010

1. Suppose X and Y are jointly continuously distributed such that both the conditional densities are positive on the cartesian product of the supports of the marginal densities, i.e.,

$$f_{X|Y}(x \mid y) > 0, \quad f_{Y|X}(y \mid x) > 0 \qquad \forall x \in S_X, \quad \forall y \in S_Y,$$

where S_X and S_Y are the supports of the marginal distributions,

$$S_X = \{x : f_X(x) > 0\}, \qquad S_Y = \{y : f_Y(y) > 0\}.$$

Show that in this case

$$f_X(x) = 1 \Bigg/ \int_{S_Y} \frac{f_{Y|X}(y \mid x)}{f_{X|Y}(x \mid y)} \, \mathrm{d}y, \tag{1}$$

$$f_X(x) = \frac{f_{X|Y}(x \mid y_0)}{f_{Y|X}(y_0 \mid x)} \Bigg/ \int_{S_X} \frac{f_{X|Y}(u \mid y_0)}{f_{Y|X}(y_0 \mid u)} \, \mathrm{d}u \tag{2}$$

for all $x \in S_X$, where in the second formula y_0 is an arbitrary point in S_Y. Hint: you obtain the first formula by integrating y out from a formula derived from the multiplication rule. To obtain the second formula, express f_Y using the first formula (exchanging the roles of X and Y) and the plug the result in the multiplication rule.

K. W. Ng calls these the **inverse Bayes formulas**: if X stands for the parameter and Y for the data, then these formulas tell how one should select the prior in order to obtain a specified posterior when the likelihood is known. In contrast, the Bayes formula tells us how the calculate the posterior, if we know the prior and the likelihood.

2. Let g be the density function of a continuously distributed random variable, whose distribution function G and quantile function are known. Now we develop a simulation method, when this distribution is truncated to the interval (c, d), where $c < d$. The pdf of the truncated distribution is proportional to the unnormalized density

$$f^*(x) = g(x)\, 1_{(c,d)}(x).$$

Develop the inverse transform method for the truncated distribution. (Start by determining the normalizing constant k such that $f(x) = f^*(x)/k$ is a density function, and then calculate the distribution function F and quantile function of the truncated distribution.)

3. (Ratio of uniforms.) Let $h \geq 0$ be an unnormalized density on the real line, i.e., $0 < \int h(x)\, \mathrm{d}x < \infty$. Define the set

$$C = \{(u, v) : 0 < u < \sqrt{h(v/u)}\},$$

and let (U, V) have the uniform distribution in C. Define $X = V/U$ and $Y = U$.

a) What is the joint pdf of X and Y?

b) Show, by marginalizing the joint pdf, that the density of X is $h(x)/k$, where $k = \int h(x)\, \mathrm{d}x$ is the normalizing constant.

4.

a) Suppose you want to simulate $N(0, 1)$ using the standard Cauchy distribution $\mathrm{Cau}(0, 1)$ as your proposal distribution in the accept–reject method. Determine the value of the majorizing constant M so that you get the greatest possible acceptance probability.

b) Show that it is not possible to simulate the standard Cauchy distribution by using $N(0, 1)$ as the proposal distribution in the accept–reject method.

5. (Smith and Gelfand, *The American Statistician*, 1992.) Suppose that we want use the accept–reject method to simulate from the posterior and that we are able to calculate the maximum likelihood estimate (MLE) $\hat{\theta}$, i.e., we know that

$$f_{Y|\Theta}(y \mid \theta) \leq f_{Y|\Theta}(y \mid \hat{\theta}) \qquad \forall \theta$$

Show that you can then use the prior as the proposal density. What is the corresponding majorizing constant M?

Often the posterior density is highly peaked while the prior is rather flat. Would this pose problems in practice?