Computational Statistics
First course exam, 1 Mar 2010.

1. We have observed $y_1, \dots, y_n$ which we model as the values of random variables Y_i, where

$$[Y_i \mid \theta] \sim \mathrm{Exp}(\theta), \qquad i = 1, \dots, n$$

independently. The prior is $\mathrm{Gam}(a, b)$ with known hyperparameters $a > 0$ and $b > 0$.

a) Give a formula for the likelihood. (2 points)

b) Recognize the posterior and give the values of its hyperparameters. (4 points)

2. Consider a Bayesian statistical model with two parameters ϕ and ψ.

a) Explain briefly (verbally or by giving formulas), what the marginal posterior of ϕ is and what the posterior full conditional of ψ is. Explain how we can represent the (joint) posterior distribution of ϕ and ψ using these two distributions.

b) Suppose we know how to simulate values from the two distributions. How can we simulate values from the joint posterior distribution?

3. We are interested in the distribution defined by the unnormalized density

$$f^*(x) = \begin{cases} \exp(-x^a), & \text{when } x > 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $a = 1.2$. We want to find the normalizing constant $c = \int f^*(x)\,\mathrm{d}x$ and the expected value EX of the distribution. Explain how we can estimate these two quantities with importance sampling.

Consider the following three candidates for the role of the instrumental distribution in importance sampling: $\mathrm{Poi}(1)$, $\mathrm{Uni}(0, 1)$ or $\mathrm{Exp}(1)$. For each of these three distributions, state whether it is a valid instrumental distribution in this problem. If you think the candidate is not valid, state the reason why.

4.

a) Consider the distribution having the pdf

$$g(x) = 3x^2\, 1_{[0,1]}(x).$$

This distribution can be simulated using the inverse transform method. How?

b) Now consider the distribution defined by the unnormalized density

$$f^*(x) = x^2\,(1 + 4\cos(x^2))\, 1_{[0,1]}(x).$$

Write an accept–reject algorithm for simulating this distribution, where you use the density g of part a as the proposal density.

Familiar distributions

Beta distribution $\mathrm{Be}(a,b)$ with parameters $a>0, b>0$ has pdf

$$\mathrm{Be}(x \mid a,b) = \frac{1}{B(a,b)}\, x^{a-1}(1-x)^{b-1}, \quad 0<x<1.$$

$B(a,b)$ is the beta function with arguments a and b,

$$B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,\mathrm{d}u = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}.$$

Cauchy distribution Cau (with location zero and scale one) has pdf

$$\mathrm{Cau}(x) = \frac{1}{\pi(1+x^2)}.$$

Exponential distribution $\mathrm{Exp}(\lambda)$ with rate $\lambda>0$ has pdf

$$\mathrm{Exp}(x \mid \lambda) = \lambda\,\mathrm{e}^{-\lambda x}, \quad x>0.$$

Gamma distribution $\mathrm{Gam}(a,b)$ with parameters $a>0, b>0$ has pdf

$$\mathrm{Gam}(x \mid a,b) = \frac{b^a}{\Gamma(a)}\, x^{a-1}\mathrm{e}^{-bx}, \quad x>0.$$

$\Gamma(a)$ is the gamma function,

$$\Gamma(a) = \int_0^\infty x^{a-1}\mathrm{e}^{-x}\,\mathrm{d}x, \qquad a>0.$$

It satisfies $\Gamma(a+1)=a\Gamma(a)$ for all $a>0$, and $\Gamma(1)=1$, from which it follows that $\Gamma(n)=(n-1)!$, when $n=1,2,3,\ldots$

Normal distribution $N(\mu,\sigma^2)$ with mean μ and variance $\sigma^2>0$ has pdf

$$N(x \mid \mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\,\exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right).$$

Student t distribution t_ν with $\nu>0$ degrees of freedom has pdf

$$t_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\,\Gamma(\nu/2)}\left(1+x^2/\nu\right)^{-(\nu+1)/2}.$$

Uniform distribution $\mathrm{Uni}(a,b)$ on the interval (a,b), where $a<b$, has pdf

$$\mathrm{Uni}(x \mid a,b) = \frac{1}{b-a}, \qquad a<x<b.$$

Binomial distribution $\mathrm{Bin}(n,p)$, n positive integer, $0\le p\le 1$, has pmf

$$\mathrm{Bin}(x \mid n,p) = \binom{n}{x} p^x(1-p)^{n-x}, \quad x=0,1,\ldots,n.$$

Poisson distribution $\mathrm{Poi}(\theta)$ with parameter $\theta>0$ has pmf

$$\mathrm{Poi}(x \mid \theta) = \mathrm{e}^{-\theta}\,\frac{\theta^x}{x!}, \quad x=0,1,2,\ldots$$