Hypercomplex numbers

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Goldsmiths' Company Mathematics Course for Teachers
22.7.2010

A family of number systems

- Real numbers
- Complex numbers
- Quaternions
- Octonions

History of complex numbers

- Italy in the 16th century
- Discovered when looking for a general solution to cubic equations

$$x^3 + ax^2 + bx + c = 0$$

- Geometrical interpretation in the 19th century

Real numbers

- Addition: $a + b$
- Subtraction: $a - b$
- Multiplication: $a \cdot b$
- Division: a/b
- Length (or norm): $|a| = \sqrt{a^2}$

Complex numbers

- $a + bi$, where $i = \sqrt{-1}$
- Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
- Subtraction: $(a + bi) - (c + di) = (a - c) + (b - c)i$
- Multiplication: $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$
- Division:

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd}{c^2 + d^2} + \frac{-ad + bc}{c^2 + d^2} i.$$

Complex numbers

- Length:

$$|a + bi| = \sqrt{a^2 + b^2}$$

- Complex conjugate:

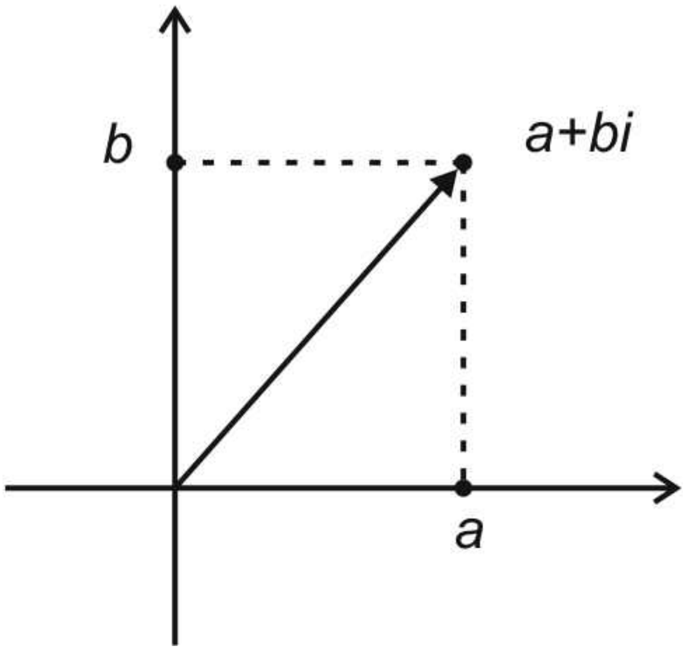
$$\overline{a + bi} = a - bi$$

- Division:

$$\frac{a + bi}{c + di} = \frac{1}{|c + di|^2} \cdot (a + bi) \cdot (\overline{c + di}).$$

Complex numbers

- Can be written as pairs of real numbers: (a, b)
- Two dimensional space $\mathbb{R}^2$
  - Addition: $(a, b) + (c, d) = (a + c, b + d)$
  - Multiplication: $(a, b) \cdot (c, d) = (ac - bd, bc + ad)$
- You can multiply and divide vectors!


b
a+bi
a

How to define multiplication?

- Would $(a, b) \cdot (c, d) = (ac, bd)$ work?

How to define multiplication?

- Multiplication: $(a, b) \cdot (c, d) = (ac, bd)$
- Division: $\dfrac{(a, b)}{(c, d)} = \left(\dfrac{a}{c}, \dfrac{b}{d}\right)$
- But now $\dfrac{(2, 1)}{(1, 0)} = \left(\dfrac{2}{1}, \dfrac{1}{0}\right)$.

Higher dimensions

- Real numbers – 1-dimensional number system
- Complex numbers – 2-dimensional number system
- Is there a 3-dimensional number system that has the same nice properties?
- Can you multiply and divide triples?
- Hamilton in 1843 Dublin

Higher dimensions

- It is impossible to divide triples.
- One needs four dimensions!

Quaternions $\mathbb{H}$

- Three square roots of -1: i, j, k
- $a + bi + cj + dk$, where a, b, c, d are real numbers
- 4-dimensional: (a, b, c, d)

Quaternions $\mathbb{H}$

- Addition:

$$(a + bi + cj + dk) + (a' + b'i + c'j + d'k)$$
$$= (a + a') + (b + b')i + (c + c')j + (d + d')k$$

- Length:

$$|a + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

- Conjugation:

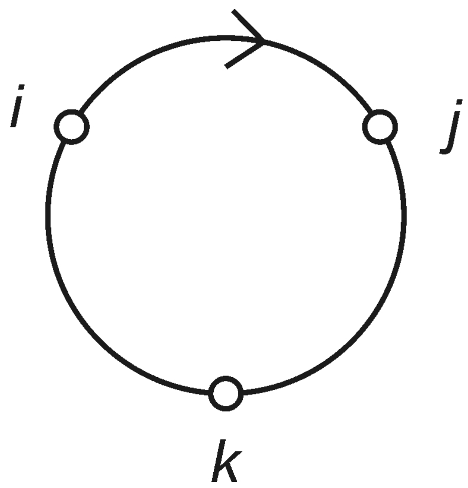
$$\overline{a + bi + cj + dk} = a - bi - cj - dk$$

Quaternion multiplication

- Multiplication: $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $jk = -i$, $ik = -j$
- Division:

$$\frac{x}{y} = \frac{1}{|x|^2} \cdot x \cdot \overline{y}$$

Quaternion multiplication



Quaternion multiplication

- $(1 + j) \cdot (-2 + k)$
- $(3 - 2i + 2j) \cdot (1 + i - 4j + k)$

Quaternion multiplication

- Quaternion multiplication is not commutative: $ij \neq ji$.

What do we have this far?

- Real numbers – 1-dimensional
- Complex numbers – 2-dimensional
- Quaternions – 4-dimensional

Higher dimensions

- Is thre a 8-dimensional number system that has the same nice properties?
- Hamilton's friend Graves in 1843

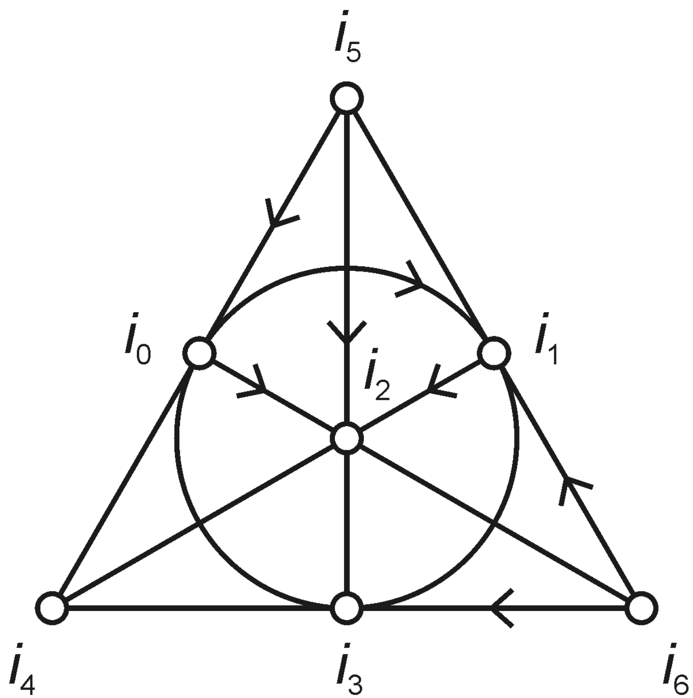
Octonions $\mathbb{O}$

- Seven square roots of -1: $i_0, i_1 \ldots, i_6$
- $a + a_0 i_0 + \cdots + a_6 i_6$,
  where $a, a_0, \ldots, a_6$ are real numbers
- 8-dimensional: $(a, a_0, \ldots, a_6)$

Octonions $\mathbb{O}$

- Addition, norm and conjugation as before

Octonion multiplication



Quaternion multiplication

- $(1 + i_0) \cdot (-2i_1 + i_3)$
- $(4 + i_2 - i_5) \cdot (2i_0 + i_2 - 3i_3)$

Octonion multiplication

- Octonion multiplication is not associative: the place of the brackets matters.
- $(i_0 i_1) i_2 = i_3 i_2 = -i_5$
- $i_0 (i_1 i_2) = i_0 i_4 = i_5$

Higher dimensions

- Real numbers – 1-dimensional
- Complex numbers – 2-dimensional
- Quaternions – 4-dimensional
- Octonions – 8-dimensional
- Is there a 16-dimensional number system that has the same nice properties?

Hurwitz's theorem (1898)

- You can divide only in dimensions 1, 2, 4 and 8.

Hurwitz's theorem

- New number systems can be built from the old ones by doubling them
  - $\mathbb{C} = \mathbb{R} + \mathbb{R}i$
  - $\mathbb{H}$ from two copies of $\mathbb{C}$
  - $\mathbb{O}$ from two copies of $\mathbb{H}$
- The dimensions are doubled too!

Hurwitz's theorem

- If we double octonions, then we get a 16-dimensional number system.
- It is not possible to divide in this number system.

My research

- I am studying structures called Lie algebras, and especially the Lie algebra E_8
- Lie algebras are important in theoretical physics, where they can be used in describing interactions between particles.
- E_8 has 248 dimensions and is quite difficult to handle.
- I have found a new simple construction for E_8 that uses octonions.

Further reading

- Paul Nahin: An imaginary tale: The story of $\sqrt{-1}$
- John Baez: The octonions (the beginning)
- Helen Joyce: Curious quaternions (Plus magazine)
- Helen Joyce: Ubiquitous octonions (Plus magazine)