Kombinatoriikka (Combinatorics)
Excercise 4

Note: the theory of rook polynomials is not covered in Cameron's book, so look up another book such as Riordan's or search for online information.

1. An international meeting needs interpreters to interpret from English to other languages. The house interpreters are Mari, Sari, Heli and Pasi and the required languages are Chinese, Russian, Swahili and Spanish. Mari speaks Spanish and Russian, Sari speaks Spanish, Heli speaks Spanish and Russian and Pasi speaks Russian, Chinese and Swahili. Draw a board representing the valid interpretes-language combinations and calulate its rook polynomial. In how many ways can three of the four languages be handled by (distinct) house interpreters?

2. A long table seats $2n$ guests, with two guests always sitting opposite of one another. To begin $2n$ guests take their seats until they all leave for the dance floor. Some k guests are then picked and snt back to k seats so that each guest is sent to his original seat or the one opposite. In how many ways can this be done?

[Instruction: define an appropriate board $B \subseteq [2n] \times [2n]$ and calculate its rook polynomial using the product rule. Use the binomial theorem to find the coefficient of x^k]

3. Let $B \subseteq [n] \times [n]$. For each $b \in B$ define

$$U(b) = \{x \in B \ : x \text{ ja } b \text{ are on the same row or column.}\}.$$

Show that if the points of the set $I \subseteq B$ all lie on the same row or column

$$R(B, x) = R(B \setminus I, x) + x(\sum_{i \in I} R(B \setminus U(i), x)).$$

4. Denote the rook polynomial of the board $[n] \times [m]$ by $R_{n,m}(x)$. Show that for all $m, n \geq 1$ the recursion

$$R_{n,m}(x) = R_{n-1,m}(x) + xmR_{n-1,m-1}(x).$$

holds.

5. Show without using the theory of hit polynomials that if $B \subseteq [n] \times [n]$ is a board, then

$$|\{\pi \in S_n : G(\pi) \cap B = \emptyset\}| = \sum_{k=0}^{n} (-1) r_k(B)(n-k)!,$$

where $G(\pi)$ is the graph $\{(i, \pi(i))\}$ of the permutation π. Hint: use PIE. Define for each $b \in B$

$$A_b = \{\pi \in S_n : b \in G(\pi)\}.$$

Now

$$\{\pi \in S_n : G(\pi) \cap B = \emptyset\} = S_n \setminus (\bigcup_{b \in B} A_b).$$