Kombinatoriikka (Combinatorics)
Excercise 4

1. Derive the following version of PIE

$$|(A_1 \cup \cdots \cup A_n)^C| = \sum_{I \subseteq [n]} (-1)^{|I|} |\bigcap_{i \in I} A_i|$$

from the form that was proved in the lectures:

$$|A_1 \cup \cdots \cup A_n| = \sum_{I \subseteq [n], I \neq \emptyset} (-1)^{|I|+1} |\bigcap_{i \in I} A_i|.$$

2. 24 students arrive for an excercise group. 9 of the students had a fever over the weekend, 10 were at the beach - as it was hot - and 5 had their exercises destroyed by a pet animal. Among the students with a fever 3 spent the weekend at the beach and 3 had their exercises destroyed by a pet. Among those students who were at the beach 2 had their exercises destroyed by a pet and one of these also had a fever.

How many students did not spend the weekend at the beach, have a fever or have their exercises destroyed by a pet?

3. Let D_n be the number of derangements i.e. fixed-point free permutations of the set $[n]$. Show by combinatorial reasoning that

$$D_n = (n-1)(D_{n-2} + D_{n-1}),$$

when $n \geq 2$.

[Hint: if $\pi : [n] \to [n]$ is a derangement then either $\pi(\pi(n)) = n$ or $\pi(\pi(n)) \neq n$. Count both cases.]

4. Around a round table are $n \geq 6$ chairs. In how many ways can one pick a set of 4 chairs, such that no three of them are adjacent?

5. Let$A_1, A_2, \ldots, A_n$ be subsets of the finite set A. Let I be a nonempty subset of the index set $[n]$. Let B be the set of points $a \in A$ such that $a \in A_i$, when $i \in I$ and $a \notin A_i$ when $i \notin I$. Show that

$$|B| = \sum_{I \subseteq J \subseteq [n]} (-1)^{|J \setminus I|} |\bigcap_{i \in J} A_i|.$$

[One may use PIE for the proof.]