Kombinatoriikka (Combinatorics)
Excercise 3

1. How many lattice paths fulfill the following conditions:

i) The path begins at $(0,-4)$ and ends in $(0,4)$

ii) The path consists of 12 steps, each of type $(1,0)$, $(-1,0)$ or $(0,1)$

iii) The path does not go via any of the points $(-2,0)$, $(-1,0)$, $(1,0)$, $(2,0)$

[*Hint: draw a picture!*]

2. Draw all ways of triangulating a regulat hexagon (6 sides) using nonintersecting diagonals, and show that all such triangulations have been drawn.

3. How many diagonals must one draw when triangulating an n-gon using nonintersecting diagonals?

4. A prison admits n inmates. The guards assess how dangerous each inmate is and assign them the numbers 1-n so that inmate i is always more dangerous than inmate j whenever $i < j$. On the first morning the prisoners are all kept in one group. On the second morning the prisoners are divided into two nonempty groups so that one group contains all the most dangerous prisoners (small numbers) and the other the least dangerous prisoners (big numbers).

From now on every morning one of the groups which still contains at least two prisoners is divided into two smaller groups with one group containing the most dangerous prisoners and the other the least dangerous prisoners from the old group. The rule is that the group containing the most dangerous possible prisoner is picked to be divided. After n days have passed each group consists of a single prisoner. In how many ways can the dividing of groups be done?

5. On next page.

5. Let $A_1, \dots, A_n$ be subsets of the set X. For each $S \subseteq [n]$ define a set

$$X_S = \bigcap_{k \in S} A_k \setminus \bigcup_{k \in S^C} A_k \subseteq X.$$

Define functions $f : \mathcal{P}([n]) \to \mathbf{N}$ and $g : \mathcal{P}([n]) \to \mathbf{N}$ as follows:

$$f(S) = |\bigcap_{k \in S} A_k|$$

$$g(S) = |X_S|.$$

Show that for all $S \subseteq [n]$ the equation

$$g(S) = f(S) - \sum_{S \subsetneq I \subset [n]} g(I)$$

holds.